1. [15 points] Suppose the xy plane is conducting and carries a current sheet of uniform but time-dependent density $\mathbf{K} = K(t)\hat{\mathbf{x}}$.

   (a) Show that the retarded vector potential above the current sheet (*i.e.*, at $z > 0$) is

$$\mathbf{A}(z,t) \;=\; \frac{\mu_0 c}{2}\hat{\mathbf{x}}\int\limits_0^\infty K(t - \tfrac{z}{c} - u)\,du. \tag{1}$$

   (b) Find the electric field $\mathbf{E}$, the magnetic field $\mathbf{B}$, and the Poynting vector above the current sheet.

   (c) Show that the current sheet radiates the EM power per unit area

$$\frac{dP}{d\text{Area}} \;=\; \frac{\mu_0 c}{2}K^2(t). \tag{2}$$

   Half of this power goes up from the current sheet, and the other half goes down.

2. A particle of mass M and charge Q is caught in a trap, where it oscillates harmonically with frequency ω around the center of the trap. As the particle oscillates, it radiates EM energy, so the amplitude of its oscillations slowly shrinks with time. Assuming the particle does not lose its energy in any other ways, how long will it take for the oscillation amplitude to reduce to $\frac{1}{2}$ of its initial value?

3. An insulated circular ring of radius b carries linear charge density $\lambda = \lambda_0 \cos\phi$. The ring rotates around its axis with a constant angular frequency ω. Take this rotation axis to be the z axis of your coordinate system, with the origin at the ring's center.

   (a) Show that the p_x and the p_y components of the ring's electric dipole moment oscillate harmonically; they have equal (real) amplitudes but different phases. Find the amplitude(s) and the phase difference for the oscillations of the $p_x(t)$ and the $p_y(t)$.

   (b) The EM fields $\mathbf{E}$ and $\mathbf{B}$ generated by the two oscillators add up linearly as vectors, but the EM power is more complicated. Find the net EM power (per unit of solid angle) $dP/d\Omega$ generated by the whole ring and explain its angular dependence.

   (c) Find the net EM power radiated by the rotating ring.

4. An FM radio station KRUD has its antenna at the top of an $h = 200$ m high tower. The antenna itself is a magnetic dipole with the vertical axis. Assume antenna's size $\ll \lambda \ll h$.

   The station's neighbors complain about excessive radiation from the station causing problems with their electronic devices, garage doors spontaneously opening and closing, and even some weird medical problems, but the city engineer who measured the broadcast power right at the bottom of the tower found the station in compliance with the city's rules. But the Neighborhood Association did not trust his report and hired you to double-check the situation.

   (a) How does the radiation intensity measured at the ground level depends on the distance from the tower?

   (b) At what distance from the tower is the radiation intensity maximal? Where should an investigator measure the radiation intensity to check if the station complies with the city's rules? And what was wrong with the city engineer's decision to measure the intensity directly under the antenna.

   (c) The KRUD broadcasts at $f = 90$ MHz; the net power emitted by its antenna is 35 kW. The city's radio emission limit is 20 $\mathrm{mW/m^2}$ at the ground level. Is KRUD in compliance with this limit?

5. [15 points] The Crab Pulsar is a young neutron star, the remnant of the supernova that blue up in 1054 AD (or rather, the light from its explosion reached Earth in 1054 AD). The remnant has mass about $2M_\odot \approx 4 \cdot 10^{30}$ kg, but its radius is only $R = 10$ km because the neutron matter is extremely dense. The Crab rotates around its axis at about 30 revolutions per second, and it has tremendously strong magnetic fields, up to $6 \cdot 10^8$ T at the surface.

   (a) Assuming the Crab's magnetic field has dipole-like profile, estimate the net magnetic moment m_0 of the Crab.

   (b) The dipole axis of the Crab's magnetic field is not aligned with its rotation axis. Instead, the dipole axis is at $\alpha \approx 45°$ angle from the rotations axis, so as the star rotates, the magnetic moment vector changes with time as

$$\mathbf{m}(t) \;=\; m_0 \sin\alpha \big[\cos(\omega t)\hat{\mathbf{x}} + \sin(\omega t)\hat{\mathbf{y}}\big] \;+\; m_0 \cos\alpha\,\hat{\mathbf{z}}. \tag{3}$$

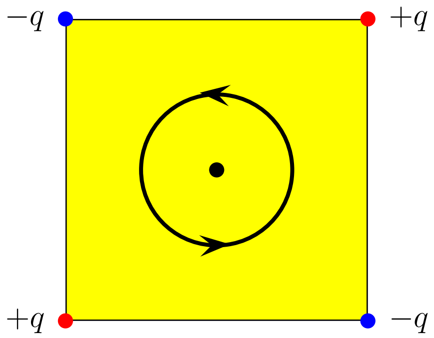
   Write down the formula for the EM radiation due to this time-dependent magnetic moment.

(c) Describe the angular distribution of the radiated EM power.

(d) Estimate the net EM power radiated by the Crab's magnetic field.

PS: The Crab is a pulsar which emits EM radiation at a wide range of frequencies, from radio waves through visible light to gamma rays. This radiation is caused by the interaction of the Crab's rotating magnetic field with the plasma surrounding the neutron star, and its net power is many orders of magnitude stronger than the radiation you calculate in this problem.

6. [15 points] Four charges $\pm q$ sit at corners of a square of size $a \times a$, which rotates with frequency Ω around it's center. That is, the rotation axis is $\perp$ to the square's plane and goes through the square's center.



(a) Find the electric quadrupole moment tensor of this system. With what frequency does it oscillate?

(b) Find the angular distribution of the EM power radiated by the rotating quadrupole.

(c) Find the net EM power radiated by the rotating quadrupole.

7. For extra credit [20 points].

In class we had derived the Larmor formula for the EM energy radiated by a (non-relativistic) point charge per unit of time. Your task in this problem is to derive similar formulae for the EM momentum and EM angular momentum radiated by the point charge per unit of time.

(a) Show that

$$\frac{d\mathbf{P}_{\rm em}}{dt} = \oint\!\!\!\!\oint d\Omega_{\mathbf{n}}\,(1-\boldsymbol{\beta}\cdot\mathbf{n})\lim_{R\to\infty}\Big(-R^2\,\overleftrightarrow{T}(R\mathbf{n})\cdot\mathbf{n}\Big), \tag{4}$$

$$\frac{d\mathbf{L}_{\rm em}}{dt} = \oiint d\Omega_{\mathbf{n}}\,(1-\boldsymbol{\beta}\cdot\mathbf{n})\lim_{R\to\infty}\Big(-R^3\mathbf{n}\times\big(\overleftrightarrow{T}(R\mathbf{n})\cdot\mathbf{n}\big)\Big), \tag{5}$$

where the integral is over a very large sphere spanned by $R\mathbf{n}$ and centered at the retarded location of the point charge.

(b) Use $c\mathbf{B} = \mathbf{n}\times\mathbf{E}$ to show that

$$-\overleftrightarrow{T}\cdot\mathbf{n} = \mathbf{n}\epsilon_0\Big(\mathbf{E}^2 - \tfrac{1}{2}(\mathbf{n}\cdot\mathbf{E})^2\Big) - \mathbf{E}\epsilon_0(\mathbf{n}\cdot\mathbf{E}). \tag{6}$$

(c) Show that in the $R\to\infty$ limit, the integrand of eq. (4) becomes

$$(1-\boldsymbol{\beta}\cdot\mathbf{n})R^2\epsilon_0\mathbf{E}_{\rm rad}^2 = \frac{Q^2\mu_0}{16\pi^2c^2}\,\frac{[\mathbf{n}\times((\mathbf{n}-\boldsymbol{\beta})\times\mathbf{a})]^2}{(1-\boldsymbol{\beta}\cdot\mathbf{n})^5}\,\mathbf{n}. \tag{7}$$

(d) Take a *careful* non-relativistic limit of this integrand by expanding it into power of β and retaining all terms proportional to $|beta^0$ and β^1 but neglecting the terms of higher-order in β. Then integrate the result over the directions of $\mathbf{n}$ to get the net EM momentum radiation rate of the moving point charge. Your answer should be

$$\frac{d\mathbf{P}_{\rm em}}{dt} = \frac{\mu_0Q^2}{6\pi c^3}\,\mathbf{a}^2\,\mathbf{v}\,. \tag{8}$$

(e) Now look at the integrand of eq. (5) and show that in the $R\to\infty$ limit it becomes

$$(1-\boldsymbol{\beta}\cdot\mathbf{n})R^3\epsilon_0(\mathbf{n}\times\mathbf{E}_{\rm rad})(-\mathbf{n}\cdot\mathbf{E}_{\rm gen.Coulomb}) = \frac{Q^2\mu_0}{16\pi^2}\,\frac{(1-\beta^2)\big(-\mathbf{n}\times(\mathbf{n}\times(\mathbf{n}-\boldsymbol{\beta})\times\mathbf{a})\big)}{(1-\boldsymbol{\beta}\cdot\mathbf{n})^4}\,. \tag{9}$$

(f) Finally, take a careful non-relativistic limit of this integrand according to the same rules as in part (d): expand in powers of β and keep all terms of order β^0 or β^1 but throw away the higher order terms. Then integrate over the directions of $\mathbf{n}$ to get the net EM angular momentum radiation rate by the moving charge. Your answer should be

$$\frac{d\mathbf{L}_{\rm em}}{dt} = \frac{\mu_0Q^2}{6\pi c}\,(\mathbf{v}\times\mathbf{a}). \tag{10}$$