1. A quantum hydrogen atom initially in the excited 2p state drops to the ground 1s state while emitting a photon. Calculate the matrix element of the electric dipole operator between these two states and hence the transition rate (in the electric dipole approximation).

   For the sake of definiteness, let the initial 2p state be $|n=2,\ell=1,m=0\rangle$ with the wave function

$$\Psi_{2p}(\mathbf{r}) \;=\; \frac{re^{-r/2a}\,\cos\theta}{\sqrt{32\pi a^5}} \tag{1}$$

   while the final 1s state $|n=1,\ell=0,m=0\rangle$ has wavefunction

$$\Psi_{1s}(\mathbf{r}) \;=\; \frac{e^{-r/a}}{\sqrt{\pi a^3}}\,; \tag{2}$$

   in both formulae a is the Bohr radius

$$a \;=\; \frac{4\pi\epsilon_0\,\hbar^2}{e^2 m_e} \;=\; \frac{\hbar}{\alpha m_e c} \;\approx\; 0.53\ \text{Å}. \tag{3}$$

   For simplicity, ignore the electron's spin.

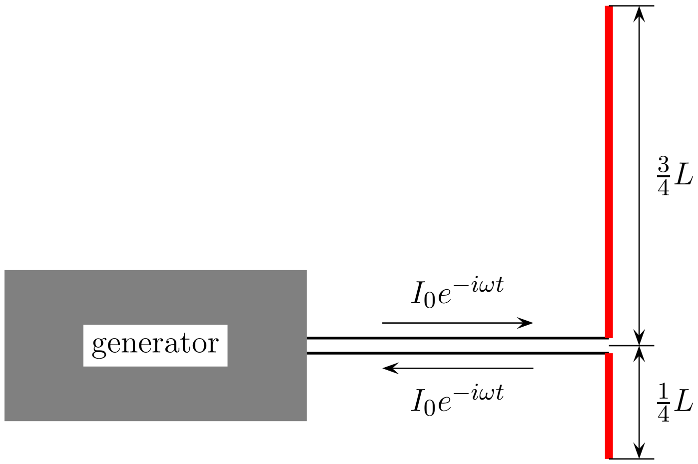
2. In a hydrogen-like atom or ion, the closest quantum analogues of circular electron orbits are the states $|n,\ell,m\rangle$ with $m=\ell=n-1$. For $n\gg 1$, the wave function of such a state is strongly peaked in a thin torus around the classical circular orbit. (This is FYI, you do not need to prove this.)

   Show that the only allowed transition from such a state to a lower-energy state is to a similar state $|n',\ell',m'\rangle$ with $m'=\ell'=n'-1$ for $n'=n-1$. Consequently, if an atom is initially in such a state with large n, then it de-excites down to the ground state through a cascade of transitions

$$\begin{gathered} |n,n-1,n-1\rangle \;\to\; |n-1,n-2,n-2\rangle \;\to\; |n-2,n-3,n-3\rangle \;\to \\ \to\;\cdots\;\to\; |3,2,2\rangle \;\to\; |2,1,1\rangle \;\to\; |1,0,0\rangle\,. \end{gathered} \tag{4}$$

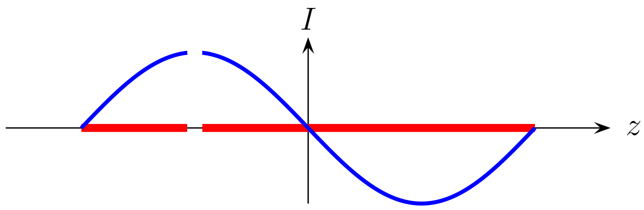
   This cascade is the quantum analogy of the classical circular orbit spiraling down to the nucleus.

3. [20 points] Consider a linear antenna that's precisely one wavelength long, $L = \lambda$. The antenna is fed at a point at distance $L/4$ from one end rather than in the middle,



For simplicity, approximate the current in the antenna by a sine wave with nodes at both ends, thus

$$I(z) \;=\; -I_0 \sin\frac{2\pi z}{L = \lambda} \tag{5}$$



Note: this sine wave is different from the current waves in the center-fed antennas, so the radiation pattern of this antenna is quite different from the $L = \lambda$ center-fed antenna discussed in class.

(a) Derive a formula for the $\mathbf{f}(\mathbf{n})$ factor for this antenna and hence for its directional power output $dP/d\Omega$. Do not use the multipole expansion.

(b) Plot the angular dependence of the directional power output $dP/d\Omega$ for this antenna as a function of the angle θ between the antenna's axis and the direction $\mathbf{n}$ towards the observer. Describe the maxima and the minima of this angular distribution.

(c) Calculate the net power emitted by the antenna and hence the antenna's radiation resistance. Note: the integral here requires special functions or numeric integration. Don't try to do it by hand but use Mathematica or equivalent software.

4. An electron is constrained to stay on the x axis, where it's subject to a harmonic potential $U(x) = \frac{1}{2}m\omega_0^2 \times x^2$. Due to Abraham–Lorentz radiation reaction force on the electron, the electron's motion is a dumped oscillation.

   (a) Write down the electron's equation of motion and show that it has 3 independent solutions: One solution is an un-physical runaway, while the other two describe a dumped oscillation.

   (b) Find the quality factor of the dumped oscillations. Assume $\omega_0\tau \ll 1$. See homework set#2, problem 3 for the definition of the quality factor.

5. A good way to avoid runaway or non-causal solutions of the equations of motion involving the Abraham–Lorentz radiation reaction force

$$\mathbf{F}_{AL} = \frac{q^2\mu_0}{6\pi c}\dot{\mathbf{a}} = m\tau\dot{\mathbf{a}} \tag{6}$$

is to replace the $m\mathbf{a}$ in this formula with just the external force $\mathbf{F}_{\text{ext}}$ without the Abraham–Lorentz force itself, hence

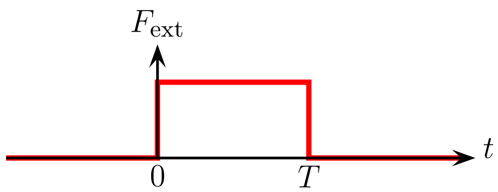
$$m\tau\dot{\mathbf{a}} \longrightarrow \tau\frac{d}{dt}\mathbf{F}_{\text{ext}}\,,$$

so Newton equation becomes

$$m\mathbf{a} = \mathbf{F}_{\text{ext}} + \tau\frac{d}{dt}\mathbf{F}_{\text{ext}}. \tag{7}$$

   (a) Apply this Newton equation (7) to the electron from the previous problem. Show that it has only two solutions, both describing dumped oscillations. Also, in the $\omega_0\tau \ll 1$ limit, you should get the same quality factor as in problem 4.

   (b) Now consider an electron subject to an external force that's constant over a finite time interval T and vanishes before or after this interval,



$$F_{\text{ext}}(t) = \begin{cases} 0 & \text{for } t < 0, \\ F_0 & \text{for } 0 < t < T, \\ 0 & \text{for } t > T. \end{cases} \tag{8}$$

   Solve the Newton equation (7) for this electron and show that the solution features neither pre-acceleration (*i.e.*, acceleration at $t < 0$) nor exponential runaway.