1. Astronaut John Blow got unlucky while walking on the Moon's surface: he stumbled and fell down, quite painfully. At precisely the same time his twin sister Mary back on Earth cried out. As it happens, an alien probe flew by the Earth–Moon system and observed both events. The probe's velocity was $v = \frac{12}{13}c$ in the direction from the Moon to the Earth. FYI, the Moon is 384,000 km away from the Earth.

   Which event — John's fall of Mary's cry — did the probe see first? And what was the time difference (in seconds, by the probe's clock) between the two events?

2. A rotation in 2D Euclidean space can be parametrized by the slope y'/x, but a more convenient parameter is the rotation angle

$$\phi = \arctan\frac{y'}{x}\,. \tag{1}$$

   By analogy, a Lorentz boost in the pseudo-Euclidean space can be parametrized by the *rapidity*

$$w = \operatorname{ar}\tanh\frac{v}{c}\,. \tag{2}$$

   (a) Express the Lorentz transform matrix in terms of the rapidity w and compare to the rotation matrix written in terms of the angle ϕ.

   Under successive rotations around the same axis, the angles ϕ add up, $\phi_{12} = \phi_1 + \phi_2$. Likewise, under successive Lorentz boosts in the same direction, the rapidities add up,

$$w_{12} = w_1 + w_2\,. \tag{3}$$

   (b) Use the relativistic velocity addition formula to prove eq. (3).

3. An inertial frame K moves relative to another inertial frame K' at velocity

$$\mathbf{v} \;=\; \beta c\big(\cos\phi\hat{\mathbf{x}} + \sin\phi\hat{\mathbf{y}}\big). \tag{4}$$

The (x, y, z) and (x', y', x') coordinate axis of the two frames are parallel to each other, and the coordinate origins coincide at $t' = t = 0$.

Write down the Lorentz transformation matrix $L^{\mu}_{\ \nu}$ between the two frames.

4. In some inertial frame K with coordinates $\mathbf{r} = (x, y, z)$ and ct, event A happens at $\mathbf{r}_A = (5, 3, 0)$ at time $ct_A = 15$ while event B happens at $\mathbf{r}_B = (10, 8, 0)$ at time $ct_B = 5$.

   (a) What is the invariant interval between events A and B?

   (b) Is there an inertial frame K' in which the two events happen simultaneously? If yes, find the velocity $\mathbf{v}'$ of that frame relative to K.

   (c) Is there an inertial frame K'' in which the two events happen at the same place? If yes, find the velocity $\mathbf{v}''$ of that frame relative to K.

   (d) Repeat parts (a) through (c) for events C and D which happen at $\mathbf{r}_C = (2, 0, 0)$, $ct_C = 1$ and $\mathbf{r}_D = (5, 0, 0)$, $ct_D = 3$.

5. The proper velocity of some object is the 4–vector

$$u^{\mu} \;=\; \frac{dx^{\mu}}{d\tau} \;=\; \big(\gamma c; \gamma v^{x}, \gamma v^{y}, \gamma v^{z}\big) \tag{5}$$

where $\mathbf{v} = (v^x, v^y, x^z)$ is the ordinary velocity vector of the object.

   (a) Give the 3 space components u^x, u^y, u^z of the proper velocity — but not the u^0 component — find the ordinary velocity $\mathbf{v}$ of the object.

   (b) A particle moves in the $\hat{\mathbf{x}}$ direction at *rapidity* w. find its proper velocity 4–vector u^{μ}.

6. [15 points] In an inertial frame K, a particle moves at speed $v = \sqrt{\frac{4}{5}}c$ in the direction at 45° between the positive x and y axes.

   (a) Before you do anything else, write down the ordinary velocity vector of this particle in (v^x, v^y, v^z) components.

   (b) Write down the proper velocity 4–vector in (u^0, u^x, u^y, u^z) components.

   Now consider a new inertial frame K' which moves relative to the K frame at speed $v[k'] = \sqrt{\frac{2}{5}}c$ in the positive x direction.

   (c) Find the proper velocity u'^μ of the particle relative to the K' frame.

   (d) As a cross-check, verify that $u'^\mu u'_\mu = c^2$.

   (e) Find the ordinary velocity 3–vector $\mathbf{v}'$ of the particle relative to the K' frame.

7. For a system of several particles, the *center of inertia frame* is the inertial frame where the net 3–momentum $\mathbf{P}_{\text{net}}$ happens to vanish. In the non-relativistic limit, this happens in the center-of-mass frame, so the center-of-inertia frame is often mis-called the center-of-mass frame.

   Suppose in some inertial frame K you have n particles with 3–momenta $\mathbf{p}_1, \ldots \mathbf{p}_n$ and relativistic energies $E_1, \ldots, E_n$. Find the velocity of the center-of-inertia frame relative to the K frame.

8. A particle of rest mass m and net energy $E = 2 \times mc^2$ collides with a similar particle at rest. After the collision, the two particles stick together.
   Find the velocity of the combined particle and its rest mass.

9. A neutral pion π^0 of rest mass $m = 135\ \text{MeV}/c^2$ has relativistic momentum $p = \frac{3}{4}mc$ in the $+\hat{\mathbf{x}}$ direction. The pion decays into two photons; one photon happens to move in the same $+\hat{\mathbf{x}}$ direction as the original pion, while the other photons moves in the opposite $-\hat{\mathbf{x}}$ direction.
   Find the relativistic energy of each photon.

10. Consider a reaction of elementary particles: a pion π^+ collides with a proton p^+ and they turn into a kaon K^+ and a Σ^+ hyperon,

$$\pi^+ \;+\; p^+ \;\longrightarrow\; K^+ \;+\; \Sigma^+. \tag{6}$$

For your information, the rest masses of the 4 particles involved in this collision are

$$M_\pi \;=\; 140\,\mathrm{MeV}/c^2, \quad M_p \;=\; 938\,\mathrm{MeV}/c^2, \quad M_K \;=\; 494\,\mathrm{MeV}/c^2, \quad M_\Sigma \;=\; 1189\,\mathrm{MeV}/c^2. \tag{7}$$

Think of the reaction (6) as an inelastic collision. The initial proton is at rest. Find the minimal energy (called the *threshold* energy) of the initial pion that would allow for the reaction (6).

Hint: use $s = (P_{\mathrm{net}} \cdot P_{\mathrm{net}})$.