1. (Postponed from set#11.)
   Consider a reaction of elementary particles: a pion π^+ collides with a proton p^+ and they turn into a kaon K^+ and a Σ^+ hyperon,
   $$\pi^+ \;+\; p^+ \;\longrightarrow\; K^+ \;+\; \Sigma^+. \tag{1}$$
   For your information, the rest masses of the 4 particles involved in this collision are
   $$M_\pi \;=\; 140\,\mathrm{MeV}/c^2,\quad M_p \;=\; 938\,\mathrm{MeV}/c^2,\quad M_K \;=\; 494\,\mathrm{MeV}/c^2,\quad M_\Sigma \;=\; 1189\,\mathrm{MeV}/c^2. \tag{2}$$
   Think of the reaction (1) as an inelastic collision. The initial proton is at rest. Find the minimal energy (called the *threshold* energy) of the initial pion that would allow for the reaction (1).
   Hint: use $s = (P_{\mathrm{net}} \cdot P_{\mathrm{net}})$.

2. Consider the *proper acceleration* 4–vector
   $$\alpha^\mu \;\stackrel{\mathrm{def}}{=}\; \frac{du^\mu}{d\tau}\,. \tag{3}$$
   (a) Spell out the components of the α^μ in terms of the ordinary acceleration $\mathbf{a}$ and velocity $\mathbf{v}$.
   (b) Check that $\alpha_\mu u^\mu = 0$ and explain why that is important.
   (c) Show that in terms of the ordinary acceleration $\mathbf{a}$,
   $$-\alpha^\mu \alpha_\mu \;=\; \gamma^6 \mathbf{a}_\parallel^2 \;+\; \gamma^4 \mathbf{a}_\perp^2 \;=\; \gamma^6 \big[\mathbf{a}^2 \;-\; (\boldsymbol{\beta} \times \mathbf{a})^2\big]. \tag{4}$$

3. Use eq. (4) and the Larmor formula (for the radiation by a non-relativistic charge) to derive the Liénard–Larmor formula (for the radiation by a relativistic charge).

4. Show that the Newton-like law for the acceleration of a charged particle in given electric+magnetic fields is

$$\gamma m \mathbf{a} \;=\; Q\Big(\mathbf{E} + \mathbf{v}\times\mathbf{B} \;-\; \boldsymbol{\beta}(\boldsymbol{\beta}\cdot\mathbf{E})\Big). \tag{5}$$

5. Consider two charges Q_A and Q_B moving relative to each other at constant velocity $\mathbf{v}$. In the inertial frame $\mathcal{S}$ of the first charge,

$$\mathbf{r}_A(t) \;\equiv\; 0 \quad\text{while}\quad \mathbf{r}_B(t) \;=\; vt\hat{\mathbf{x}} \;+\; b\hat{\mathbf{y}} \tag{6}$$

for a constant b, while in the inertial frame $\mathcal{S}'$ of the second charge

$$\mathbf{r}'_B(t') \;\equiv\; 0 \quad\text{while}\quad \mathbf{r}'_A(t') \;=\; -vt'\hat{\mathbf{x}} \;-\; b\hat{\mathbf{y}}. \tag{7}$$

(a) In the rest frame $\mathcal{S}$ of Q_A, calculate the EM force on Q_B (due to the fields generated by the Q_A) at the moment $t=0$.

(b) Now calculate the force on Q_B in the rest frame $\mathcal{S}'$ of Q_B at the moment $t'=0$.

(c) Verify that the answers to parts (a) and (b) are consistent with the Lorentz transform of the force between the two frames.

6. Consider a plane EM wave; in some inertial frame $\mathcal{S}$ the EM fields are

$$\mathbf{E}(x,y,z;t) \;=\; E_0\cos(kx-\omega t)\hat{\mathbf{y}}, \qquad \mathbf{B}(x,y,z;t) \;=\; \frac{E_0}{c}\cos(kx-\omega t)\hat{\mathbf{z}}. \tag{8}$$

Now consider the same wave in a different frame $\mathcal{S}'$ which moves relative to $\mathcal{S}$ at constant velocity $\mathbf{v}$ in the $\hat{\mathbf{x}}$ direction.

(a) Spell out the EM fields $\mathbf{E}'(x',y',z';t')$ and $\mathbf{B}'(x',y',z';t')$ in the $\mathcal{S}'$ frame as functions of its coordinates $(x',y',z';t')$.

(b) What is the frequency ω' and wavelength λ' of the wave in the $\mathcal{S}'$ frame. Do they agree with $\omega'\times\lambda' = 2\pi c = \omega\times\lambda$? If yes, explain why. If no, explain how.

(c) Find the amplitude and the intensity of the wave in the $\mathcal{S}'$ frame and their ratios to the amplitude and the intensity in the $\mathcal{S}$ frame.

(d) Young Einstein wandered what an EM wave would look like if you could run beside it at a speed of light. We cannot literally run at the speed of light, but consider the frame $\mathcal{S}'$ moving at speed v just a little slower then light. What happens to the frequency, the amplitude, and the intensity of the wave in the $v \to c$ limit?

7. [15 points] Consider a Lorentz boost of velocity $\mathbf{v}$ in the $\hat{\mathbf{x}}$ direction. Spell out how the EM energy density U, the components S^i of the Poynting vector, and the components T^{ij} of the EM stress tensor transform under such boost.

8. [20 points] Consider the EM fields $\mathbf{E}$ and $\mathbf{B}$ at some spacetime point.

(a) Show that the combination $\mathbf{E}^2 - c^2\mathbf{B}^2$ (in MKSA units) at that point is invariant under all Lorentz transforms.

(b) Show that the combination $\mathbf{E} \cdot \mathbf{B}$ is also invariant under all *continuous* Lorentz transforms.

Moreover, these are the only independent Lorentz invariant combinations. To see that, we first show that unless both $\mathbf{E} \perp \mathbf{B}$ and $|\mathbf{E}| = c|\mathbf{B}|$, — like in a plane wave, — then there is a frame where the $\mathbf{E}'$ and the $\mathbf{B}'$ vectors are parallel to each other (or one of them vanishes).

(c) Show that in the frame moving at velocity $\mathbf{v}$ such that

$$\frac{\boldsymbol{\beta}}{1+\beta^2} = \frac{\mathbf{E} \times c\mathbf{B}}{\mathbf{E}^2 + c^2\mathbf{B}^2} \qquad \text{[MKSA units]} \tag{9}$$

we have $\mathbf{E}' \times \mathbf{B}' = 0$ and hence $\mathbf{E}'$ parallel to $\mathbf{B}'$ (or one of them vanishes).

Use the coordinate system where $\mu_0\mathbf{S}\mathbf{E} \times \mathbf{B}$ points in $\hat{\mathbf{x}}$ direction. First, show that in these coordinates $T^{xx} = -U$ while $T^{xy} = T^{xz} = 0$. Then use the Lorentz transform laws of the $\mathcal{T}^{\mu\nu}$ tensor from problem 7 to find a frame where $\mathbf{S}' = 0$.

(d) Show that eq. (9) has a physical solution for $\boldsymbol{\beta}$ unless both $\mathbf{E} \perp \mathbf{B}$ and $|\mathbf{E}| = c|\mathbf{B}|$

(e) Finally, show that the EM fields $\mathbf{E}$ and $\mathbf{B}$ do not have any Lorentz-invariant combinations independent of $\mathbf{E}^2 - c^2\mathbf{B}^2$ and $\mathbf{E}\cdot\mathbf{B}$.

Note: Lorentz-invariant means invariant under both Lorentz boosts and space rotations (and hence under any combinations of boosts and rotations).

9. Consider an ideal magnetic dipole $\mathbf{m}$. In the frame $\mathcal{S}$ where this dipole is at rest at the coordinate origin, the EM potentials are.

$$\mathbf{A}(\mathbf{r}) \;=\; \frac{\mu_0}{4\pi}\,\frac{\mathbf{m}\times\mathbf{n}}{r^3}\,, \qquad V \;\equiv\; 0. \tag{10}$$

Now let's go to a different inertial frame $\mathcal{S}'$ where the dipole moves at constant velocity $\mathbf{v}$.

(a) Find the scalar potential $V'(\mathbf{r}', t')$ in this frame.

(b) Now take the non-relativistic limit $v \ll c$ of this scalar potential. Show that in this limit, V' looks like the scalar potential of the pure electric dipole

$$\mathbf{p} \;=\; \frac{\mathbf{v}\times\mathbf{m}}{c^2}\,. \tag{11}$$

10. In some frame $\mathcal{S}$, the electric and the magnetic fields are uniform, constant, and $\perp$ to each other, say

$$\mathbf{E} \;\equiv\; E\hat{\mathbf{y}}, \qquad \mathbf{B} \;\equiv\; B\hat{\mathbf{z}}. \tag{12}$$

(a) Show that if $E < cB$ then there is another frame $\mathcal{S}$ where $\mathbf{E}' = 0$. Find the velocity $\mathbf{u}$ of this frame relative to $\mathcal{S}$ and the magnetic field $\mathbf{B}'$ in the $\mathcal{S}'$ frame.

A charged particle is released from rest at the origin of the $\mathcal{S}$ frame. In the $\mathcal{S}'$ frame, the initial velocity of this particle is $\mathbf{v}_0 = -\mathbf{u}$.

(b) Describe the motion of this particle in the $\mathcal{S}'$ frame.

(c) Describe the same motion in the $\mathcal{S}$ frame.