This is an open-notes, open-book exam, so please do not waste your time copying parts of my class notes or homework solutions. If you need to use any homework result, simply reference the appropriate statement or equation and go ahead; likewise for the posted class notes, or anything else I have explained in class. Similarly, feel free to quote the textbook.

1. A thin spherical shell of radius a has uniform electric charge density $\sigma = Q/\pi a^2$.

   (a) Use electromagnetic stress tensor $\overleftrightarrow{T}$ to show that the net force between the Northern and the Southern hemispheres of this shell is related to the electric field in the equatorial plane as

   $$F = \frac{\epsilon_0}{2} \iint \mathbf{E}^2 \, d\text{area}. \tag{1}$$

   The integral here is over the whole equatorial plane, both inside and outside the shell.

   (b) Evaluate this integral and calculate the force.

2. A plane EM wave strikes a layer of plasma at incidence angle α.

   (a) Show that the total internal reflection of this wave happens for

   $$\omega < \frac{\omega_p}{\cos\alpha} \tag{2}$$

   where ω_p is the plasma frequency.

   (b) For example, consider the F layer of the Earth's ionosphere; its thickest part is about $0.05\ R_{\text{Earth}} \approx 200$ miles above the ground, so a wave emitted horizontally from the ground strikes it at $\alpha \approx 72°$. The night-time electron density in the F layer is about $10^{12}\ \text{m}^{-3}$.

   Find the highest frequency of the wave which could be totally reflected by this layer. FYI, $(e^2/\epsilon_0 m_e) \approx 3180\ \text{m}^3/\text{s}^2 \approx (2\pi \times 9\ \text{Hz})^2 \cdot \text{m}^3$.

3. Consider an attenuating plane EM wave

$$\mathbf{E} \;=\; E_0 \exp(ik_1 z - i\omega t)\exp(-k_2 z)\hat{\mathbf{x}}, \qquad \mathbf{H} \;=\; H_0 \exp(ik_1 z - i\omega t)\exp(-k_2 z)\hat{\mathbf{y}} \tag{3}$$

propagating through a conducting but non-magnetic material, $\sigma > 0$ but $\mu = 1$. Your task is to check the Poynting theorem for this wave.

(a) Calculate the time-averaged Poynting vector and its divergence $\nabla \cdot \langle \mathbf{S} \rangle$ for this wave.

(b) Relate the magnetic amplitude H_0 of the wave to the electric amplitude E_0, then show that

$$\nabla \cdot \langle \mathbf{S} \rangle \;=\; -\tfrac{1}{2}\sigma \left|E_0\right|^2 \exp(-2k_2 z). \tag{4}$$

(c) Calculate the time-averaged power loss density by the wave and verify that the wave obeys the Poynting theorem.

4. For many transparent materials, the wavelength dependence of the refraction index may be approximated by the Cauchy formula

$$n(\lambda) \;\approx\; A \;+\; \frac{B}{\lambda^2} \tag{5}$$

where $\lambda = 2\pi c/\omega$ is the *vacuum wavelength* of the light wave. For example, the dense flint glass SF10 has $A \approx 1.7280$ and $B \approx 0.01342\ \mu\text{m}^2$.

(a) Find the phase velocity and the group velocity of the cyan light with $\lambda = 0.500\ \mu$m in this flint glass.

(b) When the light pulses travel through a thick slab of this glass, they widen due to dispersion, and when they get too wide they merge up. Find the maximal rate at which the cyan light pulses can travel through $L = 1$ m of glass without merging up.