ELECTROMAGNETIC POTENTIALS

The four microscopic Maxwell equations

$$\nabla\cdot\mathbf{B} = 0, \tag{M1}$$
$$\nabla\times\mathbf{E} + \frac{\partial\mathbf{B}}{\partial t} = 0, \tag{M2}$$
$$\nabla\cdot\mathbf{E} = \frac{1}{\epsilon_0}\rho, \tag{M3}$$
$$\nabla\times\mathbf{B} - \frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t} = \mu_0\mathbf{J}, \tag{M4}$$

are often divided into two pairs: The *homogeneous Maxwell equations* (M1) and (M2) which don't depend on any charges or currents, and the *inhomogeneous Maxwell equations* (M3) and (M4) which do depend on ρ and $\mathbf{J}$. The most general solution to the two homogeneous equations obtain from an arbitrary *scalar potential* $V(\mathbf{r},t)$ and a similarly arbitrary *vector potential* $\mathbf{A}(\mathbf{r},t)$ as

$$\mathbf{B}(\mathbf{r},t) = \nabla\times\mathbf{A}(\mathbf{r},t), \qquad \mathbf{E}(\mathbf{r},t) = -\frac{\partial\mathbf{A}(\mathbf{r},t)}{\partial t} - \nabla V(\mathbf{r},t). \tag{1}$$

Indeed, $\nabla\cdot\mathbf{B} = \nabla\cdot(\nabla\times\mathbf{A}) = 0$ to satisfy eq. (M1), while

$$\frac{\partial\mathbf{B}}{\partial t} + \nabla\times\mathbf{E} = \cancel{\frac{\partial}{\partial t}(\nabla\times\mathbf{A})} - \cancel{\nabla\times\frac{\partial\mathbf{A}}{\partial t}} - \nabla\times(\nabla V) = 0 \tag{2}$$

to satisfy eq. (M2). Conversely, eq. (1) provide the most general solution to eqs. (M1) and (M2) that hold true everywhere in space and for all times. Indeed, a mathematical **Theorem** — which is a special case of the Poincaré Lemma for the differential forms — says that *a vector field whose curl vanishes everywhere in space is a gradient of some scalar field, while a vector field whose divergence vanishes everywhere in space is a curl of another vector field.* In particularly, the magnetic field obeying eq. (M1) must be a curl of some vector potential $\mathbf{A}(\mathbf{r})$. Moreover, this must be true at all times t, so a time-dependent magnetic field should be a curl of some time-dependent vector potential, according to the first eq. (1). Consequently, the Induction Law (M2) becomes

$$0 \;=\; \nabla\times\mathbf{E} \;+\; \frac{\partial\mathbf{B}}{\partial t} \;=\; \nabla\times\mathbf{E} \;+\; \frac{\partial}{\partial t}(\nabla\times\mathbf{A}) \;=\; \nabla\times\left(\mathbf{E}+\frac{\partial\mathbf{A}}{\partial t}\right): \tag{3}$$

The vector field $\mathbf{E}+\dot{\mathbf{A}}$ has zero curl, so it should be a gradient of some scalar field. Calling that scalar field $-V(\mathbf{r},t)$, we get

$$\mathbf{E}(\mathbf{r},t) \;=\; -\dot{\mathbf{A}}(\mathbf{r},t) \;-\; \nabla V(\mathbf{r},t), \tag{4}$$

precisely as in the second eq. (1).

For any given time-dependent fields $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{B}(\mathbf{r},t)$, the potentials $\mathbf{A}(\mathbf{r},t)$ and $V(\mathbf{r},t)$ are far from unique: there is a whole family of such potentials related by time-dependent *gauge transforms*

$$\mathbf{A}'(\mathbf{r},t) \;=\; \mathbf{A}(\mathbf{r},t) \;+\; \nabla\Lambda(\mathbf{r},t), \quad V'(\mathbf{r},t) \;=\; V(\mathbf{r},t) \;-\; \frac{\partial}{\partial t}\Lambda(\mathbf{r},t) \tag{5}$$

for a completely general function $\Lambda(\mathbf{r},t)$. Indeed, adding a gradient to the vector potential $\mathbf{A}$ would not affect its curl, thus

$$\mathbf{B}' \;=\; \nabla\times\big(\mathbf{A}'=\mathbf{A}+\nabla\Lambda\big) \;=\; \nabla\times\mathbf{A} \;+\; \nabla\times(\nabla\Lambda) \;=\; \mathbf{B} \;+\; \mathbf{0}, \tag{6}$$

while for the electric field, the effect of Λ cancels out from the transformed $\mathbf{A}'$ and V' potentials,

$$\begin{aligned}\mathbf{E}' \;&=\; -\frac{\partial\mathbf{A}'}{\partial t} \;-\; \nabla V' \\ &=\; -\frac{\partial\mathbf{A}}{\partial t} \;-\; \cancel{\frac{\partial}{\partial t}\nabla\Lambda} \;-\; \nabla V \;+\; \cancel{\nabla\frac{\partial\Lambda}{\partial t}} \\ &=\; -\frac{\partial\mathbf{A}}{\partial t} \;-\nabla V \;=\; \mathbf{E}.\end{aligned} \tag{7}$$

I shall return to the gauge transforms momentarily. But meanwhile, let me notice that while the potentials V and $\mathbf{A}$ and the eqs. (1) automagically solve the two homogeneous

Maxwell equations, the two inhomogeneous equations (M3) and (M4) become second-order PDEs in terms of the potentials. Specifically,

$$\frac{\rho}{\epsilon_0} = \nabla\cdot\mathbf{E} = -\nabla\cdot\frac{\partial\mathbf{A}}{\partial t} - \nabla^2 V, \tag{8}$$

$$\begin{aligned}\mu_0\mathbf{J} &= \nabla\times\mathbf{B} - \mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t} = \nabla\times(\nabla\times\mathbf{A}) - \frac{1}{c^2}\frac{\partial}{\partial t}\left(-\frac{\partial\mathbf{A}}{\partial t} - \nabla V\right)\\ &= \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A} + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{A} + \frac{1}{c^2}\nabla\left(\frac{\partial V}{\partial t}\right).\end{aligned} \tag{9}$$

For every solution of these equations, there is a whole family of physically equivalent solutions related by the gauge transforms (5). To eliminate this redundancy, we should impose an additional linear condition — called the *gauge condition* — at every spacetime point (x, y, z, t). Two most commonly used gauge conditions are:

- the *transverse gauge* $\nabla\cdot\mathbf{A}\equiv 0$, also known as the *Coulomb gauge* or as the *radiation gauge*;
- the *Landau gauge*

$$\nabla\cdot\mathbf{A} + \frac{1}{c^2}\frac{\partial V}{\partial t} = 0. \tag{10}$$

In the transverse gauge, eq. (8) reduces to the ordinary Poisson equation for the scalar potential,

$$\nabla^2 V = -\frac{\rho}{\epsilon_0}. \tag{11}$$

There are no time derivatives in this equation, only the space derivatives, so its solution is the *instantaneous* Coulomb potential of the charge density $\rho(\mathbf{r}, t)$,

$$V(\mathbf{r}, t) = \iiint \frac{\rho(\mathbf{r}', t)\, d^3\mathrm{Vol}'}{4\pi\epsilon_0\,|\mathbf{r}-\mathbf{r}'|}, \tag{12}$$

hence the name *Coulomb gauge.* Note that instantaneous propagation of the scalar potential does not mean instantaneous propagation of the electric field or any other physical quantity;

instead, the electric field propagates only at the speed of light according to the wave equations

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\begin{pmatrix}\mathbf{E}\\ \mathbf{B}\end{pmatrix} = 0 \tag{13}$$

that we have spend several weeks studying. What happens is that any change of ρ at some point $\mathbf{r}'$ leads to instantaneous changes of both scalar and vector potentials V and $\mathbf{A}$ at other points $\mathbf{r}$, but these changes cancel out from the electric field $\mathbf{E}$. But in addition to the instantaneous $\delta\mathbf{A}$ there is also a delayed $\delta\mathbf{A}$ which propagates at the speed of light, and it's that delayed piece which changes the electric field.

To see how this works, consider the vector potential $\mathbf{A}(\mathbf{r},t)$ in the transverse gauge. For $\nabla\cdot\mathbf{A}\equiv 0$, eq. (9) becomes

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\mathbf{A} = \mu_0\mathbf{J} - \frac{1}{c^2}\nabla\left(\frac{\partial V}{\partial t}\right), \tag{14}$$

where the second term on the RHS can be re-expressed in terms of the charge density ρ and hence of the current density $\mathbf{J}$. Indeed, in light of eq. (12),

$$\begin{aligned}\frac{1}{c^2}\nabla\left(\frac{\partial V(\mathbf{r},t)}{\partial t}\right) &= \epsilon_0\mu_0\,\nabla_{\mathbf{r}}\frac{\partial}{\partial t}\iiint\frac{\rho(\mathbf{r}',t)\,d^3\mathrm{Vol}'}{4\pi\epsilon_0\,|\mathbf{r}-\mathbf{r}'|}\\ &= \mu_0\,\nabla_{\mathbf{r}}\iiint\frac{d^3\mathrm{Vol}'}{4\pi|\mathbf{r}-\mathbf{r}'|}\,\frac{\partial\rho(\mathbf{r}',t)}{\partial t}\\ &\qquad\langle\!\langle\text{ by the continuity equation }\rangle\!\rangle\\ &= -\mu_0\,\nabla_{\mathbf{r}'}\iiint\frac{d^3\mathrm{Vol}'}{4\pi|\mathbf{r}-\mathbf{r}'|}\,(\nabla\cdot\mathbf{J})(\mathbf{r}',t).\end{aligned} \tag{15}$$

Thus, eq. (14) becomes a forced wave equation for the vector potential,

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\mathbf{A}(\mathbf{r},t) = \mu_0\mathbf{J}_T(\mathbf{r},t), \tag{16}$$

where

$$\mathbf{J}_T(\mathbf{r},t) = \mathbf{J}(\mathbf{r},t) + \nabla_{\mathbf{r}}\iiint\frac{d^3\mathrm{Vol}'}{4\pi|\mathbf{r}-\mathbf{r}'|}\,(\nabla\cdot\mathbf{J})(\mathbf{r}',t) \tag{17}$$

is the transverse part of the current $\mathbf{J}(\mathbf{r},t)$ or simply the *transverse current*, hence the name the *transverse gauge*.

More generally, any vector field can be written as a sum of transverse field which has zero divergence and a longitudinal field which has zero curl,

$$\mathbf{f}(\mathbf{r}) \;=\; \mathbf{f}_T(\mathbf{r}) \;+\; \mathbf{f}_L(\mathbf{r}), \quad \nabla\cdot\mathbf{f}_T \;\equiv\; 0, \quad \nabla\times\mathbf{f}_L \;\equiv\; 0. \tag{18}$$

The simplest way to see how this works is via the Fourier transform

$$\mathbf{f}(\mathbf{r}) \;=\; \iiint \frac{d^3\mathbf{k}}{(2\pi)^3}\, e^{i\mathbf{k}\cdot\mathbf{r}}\,\tilde{\mathbf{f}}(\mathbf{k}), \quad \tilde{\mathbf{f}}(\mathbf{k}) \;=\; \iiint d^3\mathbf{r}\, e^{-i\mathbf{k}\cdot\mathbf{r}}\,\mathbf{f}(\mathbf{r}). \tag{19}$$

The Fourier transform of the curl and the divergence of $\mathbf{f}$ are simply $i\mathbf{k}\cdot\tilde{\mathbf{f}}(\mathbf{k})$ and $i\mathbf{k}\times\tilde{\mathbf{f}}(\mathbf{k})$, so the decomposition into the transverse and longitudinal pieces amounts to splitting the $\tilde{\mathbf{f}}(\mathbf{k})$ vector into the parts perpendicular and parallel to $\mathbf{k}$,

$$\tilde{\mathbf{f}}_L(\mathbf{k}) \;=\; \frac{\mathbf{k}\cdot\tilde{\mathbf{f}}(\mathbf{k})}{\mathbf{k}^2}\,\mathbf{k}, \quad \tilde{\mathbf{f}}_T(\mathbf{k}) \;=\; \tilde{\mathbf{f}}(\mathbf{k}) \;-\; \tilde{\mathbf{f}}_L(\mathbf{k}). \tag{20}$$

Fourier transforming back to the $\mathbf{f}(\mathbf{r})$, we find

$$\mathbf{f}_L(\mathbf{r}) \;=\; (i\nabla)\,\frac{-1}{\nabla^2}\,(i\nabla\cdot\mathbf{f}) \;=\; -\nabla_{\mathbf{r}}\iiint \frac{d^3\mathrm{Vol}'}{4\pi|\mathbf{r}-\mathbf{r}'|}\,(\nabla\cdot\mathbf{f})(\mathbf{r}') \tag{21}$$

and hence

$$\mathbf{f}_T(\mathbf{r}) \;=\; \mathbf{f}(\mathbf{r}) \;+\nabla_{\mathbf{r}}\iiint \frac{d^3\mathrm{Vol}'}{4\pi|\mathbf{r}-\mathbf{r}'|}\,(\nabla\cdot\mathbf{f})(\mathbf{r}'). \tag{22}$$

Comparing this formula to eq (17), we see that $\mathbf{J}_T(\mathbf{r},t)$ is indeed the transverse part of the electric current density at time t.

Note that extracting the transverse component of the electric current is non-local in the coordinate space; instead, the $\mathbf{J}_T(\mathbf{r},t)$ at any particular point $\mathbf{r}$ and time t depends on $\mathbf{J}(\mathbf{r}',t)$ at other points $\mathbf{r}'$ but the same time t. Consequently, the vector potential $\mathbf{A}(\mathbf{r},t)$ which obeys the forced wave equation (16) contains an instantaneously propagating piece due to $\mathbf{J}_T(\mathbf{r},t)$ at the same $(\mathbf{r},t)$, which is in turn instantaneously affected by the $\mathbf{J}(\mathbf{r}',t)$ at some other location $\mathbf{r}'$. Fortunately, one can show that this instantaneous piece is purely longitudinal, so it does not affect the magnetic field while its contribution to the electric field cancels against the instantaneous scalar potential. However, this would be a graduate-level exercise, so let me kip it in this undergraduate class.

The transverse gauge is commonly used in Quantum Electrodynamics (QED), which has to be formulated in terms of the potentials to allow *local* interaction with the Dirac field of the electrons. It is particularly convenient for calculating the EM radiation by the atoms, hence yet another name, the *radiation gauge.* Even classically, this gauge convenient for the EM radiation by small antennas, since the gradient of the scalar potential (12) decreases with distance as $1/r^2$ or faster, so it affects the electric field only at the relatively short distances from the antenna. In the radiation zone further away from the antenna, all we need is the vector potential, thus

$$\mathbf{B} \;=\; \nabla\times\mathbf{A}, \quad \mathbf{E} \;\approx\; -\frac{\partial}{\partial t}\mathbf{A}. \tag{23}$$

Now consider the **Landau gauge** (10). Writing eq. (9) for the vector potential as

$$\mu_0\mathbf{J} \;=\; \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\mathbf{A} \;+\; \nabla\left((\nabla\cdot\mathbf{A}) + \frac{1}{c^2}\frac{\partial V}{\partial t}\right), \tag{24}$$

we see that in the Landau gauge it becomes simply

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\mathbf{A}(\mathbf{r},t) \;=\; \mu_0\mathbf{J}(\mathbf{r},t). \tag{25}$$

Likewise, eq. (8) for the scalar potential — which me may write as

$$\begin{aligned}\frac{\rho}{\epsilon_0} &= -\frac{\partial}{\partial t}(\nabla\cdot\mathbf{A}) \;-\; \nabla^2 V\\ &= \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)V \;-\; \frac{\partial}{\partial t}\left(\nabla\cdot\mathbf{A} + \frac{1}{c^2}\frac{\partial V}{\partial t}\right),\end{aligned} \tag{26}$$

— in the Landau gauge becomes simply

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)V(\mathbf{r},t) \;=\; \frac{1}{\epsilon_0}\,\rho(\mathbf{r},t). \tag{27}$$

The Landau gauge is convenient for the manifestly relativistic treatment of electrodynamics. Indeed, the differential operator involved in both eqs. (25) and (27),

$$\Box \;=\; \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \;-\; \nabla^2 \;=\; \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \;-\; \frac{\partial^2}{\partial x_1^2} \;-\; \frac{\partial^2}{\partial x_2^2} \;-\; \frac{\partial^2}{\partial x_3^2} \tag{28}$$

— called the *d'Alembert operator* or simply the *d'Alembertian* — is invariant under Lorentz transformations of space and time coordinates between differently moving frames of reference.

Also, the similarity between the equations (25) and (27) allows combining the scalar and the vector potentials into a 4–vector potential, while the charge density and the current density are combined into another 4–vector. We shall return to this issue later in class, probably in April.

The Landau gauge condition does not completely fix the gauge. Indeed, let the potentials $\mathbf{A}(\mathbf{r},t)$ and $V(\mathbf{r},t)$ obey

$$\nabla\cdot\mathbf{A}\;+\;\frac{1}{c^2}\frac{\partial V}{\partial t}\;=\;0. \tag{10}$$

Then for any $\Lambda(\mathbf{r},t)$ which satisfy the free wave equation $\Box\Lambda(\mathbf{r},t)=0$, the transformed potentials

$$\mathbf{A}'\;=\;\mathbf{A}\;+\;\nabla\Lambda,\quad V'\;=\;V\;-\;\frac{\partial\Lambda}{\partial t}\,, \tag{29}$$

also obey the Landau gauge condition,

$$\nabla\cdot\mathbf{A}'\;+\;\frac{1}{c^2}\frac{\partial V'}{\partial t}\;=\;\nabla\cdot\mathbf{A}\;+\;\frac{1}{c^2}\frac{\partial V}{\partial t}\;+\;\left(\nabla^2\;-\;\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\Lambda\;=\;0. \tag{30}$$

To eliminate this residual ambiguity of the potentials, we need an additional rule such as causality. In general, *causality* requires that physical quantities such as electric or magnetic fields $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{B}(\mathbf{r},t)$ should not be affected by the charges or currents at later times $t'>t$, — today's fields cannot be affected by tomorrow's charges and currents. In principle, the potentials $\mathbf{A}$ and V do not have to be causal, as long as the fields $\mathbf{E}$ and $\mathbf{B}$ are causal. However, we may require the potentials to be causal as an additional gauge-fixing constraint to eliminate the residual ambiguity of the Landau gauge.

Solving the Wave Equations

Consider the wave equation

$$\Box\Psi(x,y,z,t)\;=\;S(x,y,z,t) \tag{31}$$

for some continuously distributed source $S(x,y,z,t)$. The causal solution to this equation is

$$\Psi(\mathbf{r},t)\;=\;\iiint\frac{d^3\mathrm{Vol}'}{4\pi\mathcal{R}}\,S(\mathbf{r}',t_{\mathrm{ret}}) \tag{32}$$

where

$$\mathcal{R} \stackrel{\text{def}}{=} \left|\mathbf{r}-\mathbf{r}'\right| \quad \text{and} \quad t_{\text{ret}} \stackrel{\text{def}}{=} t - \frac{\mathcal{R}}{c}. \tag{33}$$

Note the source $S(\mathbf{r}', t_{\text{ret}})$ is evaluated at the retarded time t_{ret}, which is earlier than the time t when we measure the wave $\Psi(\mathbf{r}, t)$ by precisely the time interval $\mathcal{R}/c$ it takes the wave to propagate from the source point $\mathbf{r}'$ to the observation point $\mathbf{r}$. That's why the solution (32) not only obeys the wave equation (31) but is also causal: the wave at time t depends on the source at earlier times t_{ret} but not at any later times.

Without this causality condition, eq. (31) would have other solutions, for example the advanced solution

$$\Psi(\mathbf{r}, t) = \iiint \frac{d^3\text{Vol}'}{4\pi\mathcal{R}}\, S(\mathbf{r}', t_{\text{adv}}) \qquad \text{for} \quad t_{\text{adv}} = t + \frac{\mathcal{R}}{c}, \tag{34}$$

but the only causal solution is (32). I am not going to prove this uniqueness in class — alas, the math involved in this proof is beyond the undergraduate level. Instead, let me prove that (32) is indeed a solution to the wave eq. (31). Note that the integration range in eq. (32) is the whole 3D space; this range is the same for all observation points and times $(\mathbf{r}, t)$, so we may put the derivatives WRT (x, y, z, t) inside the integral, thus

$$\Box\Psi(\mathbf{r}, t) = \iiint d^3\text{Vol}'\, \Box\left(\frac{S(\mathbf{r}', t_{\text{ret}})}{4\pi\mathcal{R}}\right). \tag{35}$$

Inside the integral, the space-derivative part $-\nabla^2$ of the D'alembertian is made from derivatives WRT (x, y, z) while the source S is taken at (x', y', z'), which naively suggests $\nabla_{\mathbf{r}} S(\mathbf{r}') = 0$. However, the source S is also time-dependent; moreover, it's evaluated at the retarded time t_{ret} that depends on the distance $\mathcal{R}$ from $\mathbf{r}$ to $\mathbf{r}'$. Therefore,

$$\nabla_{\mathbf{r}} S(\mathbf{r}', t_{\text{ret}}) = \dot{S}(\mathbf{r}', t_{\text{ret}}) * \nabla_{\mathbf{r}} t_{\text{ret}} \tag{36}$$

where $\dot{S} \stackrel{\text{def}}{=} \partial S/\partial t$ while

$$\nabla_{\mathbf{r}} t_{\text{ret}} = \nabla_{\mathbf{r}} \frac{-|\mathbf{r}-\mathbf{r}'|}{c} = -\frac{\mathbf{n}}{c} \quad \text{for} \quad \mathbf{n} = \frac{\mathbf{r}-\mathbf{r}'}{\mathcal{R}}, \tag{37}$$

hence

$$\nabla_{\mathbf{r}} S(\mathbf{r}', t_{\rm ret}) \;=\; -\frac{\mathbf{n}}{c}\,\dot{S}(\mathbf{r}', t). \tag{38}$$

Likewise,

$$\nabla_{\mathbf{r}} \dot{S}(\mathbf{r}', t_{\rm ret}) \;=\; -\frac{\mathbf{n}}{c}\ddot{S}(\mathbf{r}', t_{\rm ret}), \tag{39}$$

hence

$$\nabla^2_{\mathbf{r}} S \;=\; \nabla_{\mathbf{r}}\cdot\left(-\frac{\mathbf{n}}{c}\dot{S}\right) \;=\; -(\nabla_{\mathbf{r}}\cdot\mathbf{n})\frac{\dot{S}}{c} - \frac{\mathbf{n}}{c}\cdot\nabla_{\mathbf{r}}\dot{S} \;=\; -\frac{2}{\mathcal{R}}\frac{\dot{S}}{c} + \frac{\mathbf{n}}{c}\cdot\frac{\mathbf{n}}{c}\ddot{S} \;=\; -\frac{2}{\mathcal{R}c}\dot{S} + \frac{1}{c^2}\ddot{S}, \tag{40}$$

and therefore

$$\begin{aligned}
\nabla^2_{\mathbf{r}}\left(\frac{S}{\mathcal{R}}\right) &= \frac{1}{\mathcal{R}}\nabla^2_{\mathbf{r}} S \;+\; 2\left(\nabla_{\mathbf{r}}\frac{1}{\mathcal{R}}\right)\cdot\nabla_{\mathbf{r}} S \;+\; \left(\nabla^2_{\mathbf{r}}\frac{1}{\mathcal{R}}\right) S \\
&= \frac{1}{\mathcal{R}}\left(\frac{1}{c^2}\ddot{S} - \frac{2}{c\mathcal{R}}\dot{S}\right) - \frac{2\mathbf{n}}{\mathcal{R}^2}\cdot\left(-\frac{\mathbf{n}}{c}\dot{S}\right) - 4\pi\delta^{(3)}(\mathbf{r}-\mathbf{r}') * S \\
&= \frac{1}{c^2\mathcal{R}}\ddot{S} - \cancel{\frac{2}{c\mathcal{R}^2}}\dot{S} + \cancel{\frac{2}{c\mathcal{R}^2}}\dot{S} - 4\pi\delta^{(3)}(\mathbf{r}-\mathbf{r}') * S \\
&= \frac{1}{c^2\mathcal{R}}\ddot{S} - 4\pi\delta^{(3)}(\mathbf{r}-\mathbf{r}') * S.
\end{aligned} \tag{41}$$

On the other hand, for any fixed $\mathbf{r}$ and $\mathbf{r}'$, the partial time derivative WRT t is the same as partial time derivative WRT $t_{\rm ret}$,

$$\frac{\partial S(\mathbf{r}', t_{\rm ret})}{\partial t} \;=\; \frac{\partial S(\mathbf{r}', t_{\rm ret})}{\partial t_{\rm ret}} \;=\; \dot{S}(\mathbf{r}', t_{\rm ret}) \tag{42}$$

and likewise

$$\frac{\partial^2 S(\mathbf{r}', t_{\rm ret})}{\partial t^2} \;=\; \frac{\partial^2 S(\mathbf{r}', t_{\rm ret})}{\partial t^2_{\rm ret}} \;=\; \ddot{S}(\mathbf{r}', t_{\rm ret}). \tag{43}$$

Consequently,

$$\begin{aligned}
\Box_{(\mathbf{r},t)}\left(\frac{S(\mathbf{r}', t_{\rm ret})}{\mathcal{R}}\right) &= \frac{1}{c^2\mathcal{R}}\frac{\partial^2 S(\mathbf{r}', t_{\rm ret})}{\partial t^2} \;-\; \nabla^2_{\mathbf{r}}\left(\frac{S}{\mathcal{R}}\right) \\
&= \cancel{\frac{1}{c^2\mathcal{R}}\ddot{S}(\mathbf{r}', t_{\rm ret})} \;-\; \cancel{\frac{1}{c^2\mathcal{R}}\ddot{S}(\mathbf{r}', t_{\rm ret})} + 4\pi\delta^{(3)}(\mathbf{r}-\mathbf{r}') * S(\mathbf{r}', t_{\rm ret}) \\
&= 4\pi\delta^{(3)}(\mathbf{r}-\mathbf{r}') * S(\mathbf{r}', t_{\rm ret})
\end{aligned} \tag{44}$$

and therefore

$$\begin{aligned}\Box\Psi(\mathbf{r},t) &= \iiint d^3\mathrm{Vol}'\,\Box\left(\frac{S(\mathbf{r}',t_{\mathrm{ret}})}{4\pi\mathcal{R}}\right)\\ &= \iiint d^3\mathrm{Vol}'\,\delta^{(3)}(\mathbf{r}-\mathbf{r}')*S(\mathbf{r}',t_{\mathrm{ret}})\\ &= S(\mathbf{r},t).\end{aligned}\tag{45}$$

Thus, eq. (32) is indeed a solution to the forced wave equation $\Box\Psi = S$, *quod erat demonstrandum.*

Coming back from the abstract math to the electromagnetic potential, we may immediately apply eq. (32) to the equations

$$\Box V(\mathbf{r},t) = \frac{1}{\epsilon_0}\rho(\mathbf{r},t), \tag{27}$$

$$\Box\mathbf{A}(\mathbf{r},t) = \mu_0\mathbf{J}(\mathbf{r},t), \tag{25}$$

for the EM potentials in the Landau gauge. Thus, for any continuous charge and current distributions $\rho(\mathbf{r},t)$ and $\mathbf{J}(\mathbf{r},t)$, the EM potentials in the Landau gauge are

$$V(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0}\iiint\frac{d^3\mathrm{Vol}'}{\mathcal{R}}\,\rho(\mathbf{r}',t_{\mathrm{ret}}), \tag{46}$$

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\iiint\frac{d^3\mathrm{Vol}'}{\mathcal{R}}\,\mathbf{J}(\mathbf{r}',t_{\mathrm{ret}}). \tag{47}$$

Example

Take an infinitely long wire along the z axis carrying a time-dependent current

$$I(t) = \begin{cases} 0 & \text{for } t<0,\\ I_0 & \text{for } t\geq 0.\end{cases} \tag{48}$$

That is, not current for $t<0$, then at $t=0$ we abruptly turn on the current I_0 and keep it steady for all subsequent times $t>0$. Let's find the potentials $V(\mathbf{r},t)$ and $\mathbf{A}(\mathbf{r},t)$ and hence the EM fields for this current.

First of all, we presume there are no charges but only the current $I(t)$, thus $\rho(\mathbf{r},t) \equiv 0$ and therefore $V(\mathbf{r},t) \equiv 0$. As to the vector potential, plugging

$$\mathbf{J}(x,y,z,t) \;=\; I(t)\delta(x)\delta(y)\hat{\mathbf{z}} \tag{49}$$

into eq. (47) we arrive at

$$\mathbf{A}(\mathbf{r},t) \;=\; \frac{\mu_0\hat{\mathbf{z}}}{4\pi}\int dz'\,\frac{I(t_{\rm ret})}{\mathcal{R}}\,, \tag{50}$$

where in cylindrical coordinates (s,ϕ,z) for the $\mathbf{r}$

$$\mathcal{R} \;=\; \sqrt{s^2+(z-z')^2} \quad\Longrightarrow\quad ct_{\rm ret} \;=\; ct \;-\; \sqrt{s^2+(z-z')^2}. \tag{51}$$

The current $I(t)$ is turned off for $t_{\rm ret}<0$ and turned on for $t_{\rm ret}\geq 0$. In terms of the observer time t and the cylindrical coordinates, $t_{\rm ret}>0$ means

$$ct \;\geq\; \sqrt{s^2+(z-z')^2} \quad\Longrightarrow\quad (ct)^2 \;\geq\; s^2 \;+\; (z-z')^2 \quad \langle\!\langle\, \text{and also } t\geq 0\,\rangle\!\rangle, \tag{52}$$

or equivalently

$$ct \;\geq\; s \quad \textbf{and} \quad |z-z'| \;\leq\; \sqrt{(ct)^2-s^2}\,. \tag{53}$$

Therefore, for $s>ct$ we have $\mathbf{A}=0$ while for $s<ct$

$$\mathbf{A}(s,\phi,z,t) \;=\; \frac{\mu_0 I_0\hat{\mathbf{z}}}{4\pi}\int\limits_{z-b}^{z+b}\frac{dz'}{\mathcal{R}} \tag{54}$$

where $b=\sqrt{(ct)^2-s^2}$. Evaluating the integral here, we find

$$\int\limits_{z-b}^{z+b}\frac{dz'}{\mathcal{R}} \;=\; 2\ln\frac{ct+b}{s} \;=\; 2\ln\left(\frac{ct+\sqrt{(ct)^2-s^2}}{s}\right) \tag{55}$$

and therefore

$$\mathbf{A}(s,t) \;=\; \frac{\mu_0 I_0\hat{\mathbf{z}}}{2\pi}\,\ln\left(\frac{ct+\sqrt{(ct)^2-s^2}}{s}\right). \tag{56}$$

Given this vector potential, the electric field obtains as

$$\mathbf{E}(s,t) \;=\; -\frac{\partial\mathbf{A}}{\partial t} \;=\; -\frac{\mu_0 I_0 \hat{\mathbf{z}}}{2\pi}\,\frac{c}{\sqrt{(ct)^2-s^2}}\,, \tag{57}$$

but only for $ct > s$, otherwise $\mathbf{E}=0$. Likewise, the magnetic field is absent for $ct < s$ while for $ct > s$ it becomes

$$\mathbf{B}(s,t) \;=\; \nabla\times\mathbf{A} \;=\; -\frac{\partial A_z}{\partial s}\,\hat{\boldsymbol{\phi}} \;=\; +\frac{\mu_0 I_0 \hat{\boldsymbol{\phi}}}{2\pi s}\,\frac{ct}{\sqrt{(ct)^2-s^2}}\,. \tag{58}$$

As a cross-check on our answers, let's verify that at the late times $t\to\infty$, the **E** and **B** fields asymptote to their static values. Thus,

$$\mathbf{E}[\text{from eq. (57)}] \;\xrightarrow[ct\gg s]{}\; -\frac{\mu_0 I_0 \hat{\mathbf{z}}}{2\pi t} \;\xrightarrow[t\to\infty]{}\; 0 \tag{59}$$

which agrees with no electrostatic fields since there are no charges, only the current, while

$$\mathbf{B}[\text{from eq. (58)}] \;\xrightarrow[ct\gg s]{}\; \frac{\mu_0 I_0 \hat{\boldsymbol{\phi}}}{2\pi s} \tag{60}$$

which is the indeed the magnetostatic field of the steady current I_0.

Jefimenko Equations

Given the retarded potentials

$$V(\mathbf{r},t) \;=\; \frac{1}{4\pi\epsilon_0}\iiint\frac{d^3\mathrm{Vol}'}{\mathcal{R}}\,\rho(\mathbf{r}',t_{\mathrm{ret}}), \tag{46}$$

$$\mathbf{A}(\mathbf{r},t) \;=\; \frac{\mu_0}{4\pi}\iiint\frac{d^3\mathrm{Vol}'}{\mathcal{R}}\,\mathbf{J}(\mathbf{r}',t_{\mathrm{ret}}), \tag{47}$$

deriving the electric and the magnetic fields

$$\mathbf{B} = \nabla\times\mathbf{A}, \quad \mathbf{E} = -\frac{\partial\mathbf{A}}{\partial t} - \nabla V \tag{61}$$

seems to be the matter of straightforward calculus. For example,

$$\begin{aligned}
\mathbf{B}(\mathbf{r},t) &= \frac{\mu_0}{4\pi}\nabla_{\mathbf{r}}\times\iiint\frac{d^3\mathrm{Vol}'}{\mathcal{R}}\,\mathbf{J}(\mathbf{r}',t_{\mathrm{ret}}) \\
&= \frac{\mu_0}{4\pi}\iiint d^3\mathrm{Vol}'\,\nabla_{\mathbf{r}}\times\left(\frac{\mathbf{J}(\mathbf{r}',t_{\mathrm{ret}})}{\mathcal{R}}\right) \\
&= \frac{\mu_0}{4\pi}\iiint d^3\mathrm{Vol}'\left(\frac{1}{\mathcal{R}}\,\nabla_{\mathbf{r}}\times\mathbf{J}(\mathbf{r}',t_{\mathrm{ret}}) + \left(\nabla_{\mathbf{r}}\frac{1}{\mathcal{R}} = \frac{-\mathbf{n}}{\mathcal{R}^2}\right)\times\mathbf{J}(\mathbf{r}',t_{\mathrm{ret}})\right).
\end{aligned} \tag{62}$$

Note that the current here is evaluated at the point $\mathbf{r}'$ rather than $\mathbf{r}$, so naively $\nabla_{\mathbf{r}}\times\mathbf{J}(\mathbf{r}',t) = 0$. However, the current is retarded: it's evaluated at the time t_{ret} rather than the observer time t, and the retardation depends on the distance $\mathcal{R} = |\mathbf{r}-\mathbf{r}'|$. Consequently,

$$\frac{\partial}{\partial r_i}J_j(\mathbf{r}',t_{\mathrm{ret}}) = \frac{\partial t_{\mathrm{ret}}}{\partial r_i} * \dot{J}_j(\mathbf{r}',t_{\mathrm{ret}}) = -\frac{n_i}{c} * \dot{J}_j(\mathbf{r}',t_{\mathrm{ret}}), \tag{63}$$

and in particular

$$\nabla_{\mathbf{r}}\times\mathbf{J}(\mathbf{r}',t_{\mathrm{ret}}) = -\frac{\mathbf{n}}{c}\times\dot{\mathbf{J}}(\mathbf{r}',t_{\mathrm{ret}}). \tag{64}$$

Plugging this curl into eq. (62), we arrive at

$$\mathbf{B}(\mathbf{r},t) = -\frac{\mu_0}{4\pi}\iiint d^3\mathrm{Vol}'\,\mathbf{n}\times\left(\frac{1}{c\mathcal{R}}\,\dot{\mathbf{J}}(\mathbf{r}',t_{\mathrm{ret}}) + \frac{1}{\mathcal{R}^2}\,\mathbf{J}(\mathbf{r}',t_{\mathrm{ret}})\right). \tag{65}$$

This is the *Jefimenko equation*, named after Oleg D. Jefimenko who derived it in 1966. Note two differences between the Jefimenko's magnetic field (65) and the quasi-static Biot–Savart–Laplace approximation

$$\mathbf{B}(\mathbf{r},t) \approx -\frac{\mu_0}{4\pi}\iiint d^3\mathrm{Vol}'\,\frac{\mathbf{n}}{\mathcal{R}^2}\times\mathbf{J}(\mathbf{r}',t). \tag{66}$$

First, in the Jefimenko equation the current is evaluated at the retarded time $t_{\mathrm{ret}} = t - \mathcal{R}/c$ rather than at the observer time t. Second, in the Jefimenko equation, the magnetic field depends not only on the current but also on its time derivative $\dot{\mathbf{J}}$. Suppose the current changes on the time scale τ,

$$\dot{\mathbf{J}} \sim \frac{1}{\tau}\mathbf{J}. \tag{67}$$

Then the relative contribution of the Jefimenko's $\dot{\mathbf{J}}$ term versus the quasi-static $\mathbf{J}$ term is

$$\frac{\dot{\mathbf{J}}\ \text{term}}{\mathbf{J}\ \text{term}} \sim \frac{1}{c\mathcal{R}\tau} \Big/ \frac{1}{\mathcal{R}^2} = \frac{\mathcal{R}}{c\tau}: \tag{68}$$

For relatively short distances $\mathcal{R} \ll c\tau$ from the current, the Jefimenko's term is small, and we may use the quasi-static approximation. But at long distances $\mathcal{R} \gg c\tau$, the Jefimenko's term becomes dominant. Consequently, at long distances, the magnetic field scales with distance as $\mathbf{B} \propto 1/\mathcal{R}$ instead of the quasi-static behavior $\mathbf{B} \propto 1/\mathcal{R}^2$.

Now let's calculate the electric field:

$$\begin{aligned}\mathbf{E}(\mathbf{r},t) &= -\frac{\mu_0}{4\pi}\frac{\partial}{\partial t}\iiint\frac{d^3\mathrm{Vol}'}{\mathcal{R}}\,\mathbf{J}(\mathbf{r}',t_{\mathrm{ret}}) - \frac{1}{4\pi\epsilon_0}\nabla_{\mathbf{r}}\iiint\frac{d^3\mathrm{Vol}'}{\mathcal{R}}\,\rho(\mathbf{r}',t_{\mathrm{ret}}) \\ &= -\frac{1}{4\pi\epsilon_0}\iiint d^3\mathrm{Vol}'\left(\mu_0\epsilon_0\frac{\partial}{\partial t}\left(\frac{\mathbf{J}(\mathbf{r}',t_{\mathrm{ret}})}{\mathcal{R}}\right) + \nabla_{\mathbf{r}}\left(\frac{\rho(\mathbf{r}',t_{\mathrm{ret}})}{\mathcal{R}}\right)\right),\end{aligned} \tag{69}$$

where $\mu_0\epsilon_0 = 1/c^2$,

$$\frac{\partial}{\partial t}\left(\frac{\mathbf{J}(\mathbf{r}',t_{\mathrm{ret}})}{\mathcal{R}}\right) = \frac{\dot{\mathbf{J}}(\mathbf{r}',t_{\mathrm{ret}})}{\mathcal{R}}, \tag{70}$$

but

$$\nabla_{\mathbf{r}}\left(\frac{\rho(\mathbf{r}',t_{\mathrm{ret}})}{\mathcal{R}}\right) = \frac{1}{\mathcal{R}}\nabla_{\mathbf{r}}\rho(\mathbf{r}',t_{\mathrm{ret}}) - \frac{\mathbf{n}}{\mathcal{R}^2}\rho(\mathbf{r}',t_{\mathrm{ret}}) = -\frac{\mathbf{n}}{c\mathcal{R}}\dot{\rho}(\mathbf{r}',t_{\mathrm{ret}}) - \frac{\mathbf{n}}{\mathcal{R}^2}\rho(\mathbf{r}',t_{\mathrm{ret}}). \tag{71}$$

Plugging all these formulae into eq. (69), we arrive at the *Jefimenko equation for the electric field,*

$$\mathbf{E}(\mathbf{r},t) = +\frac{1}{4\pi\epsilon_0}\iiint d^3\mathrm{Vol}'\left(\frac{\mathbf{n}}{\mathcal{R}^2}\rho(\mathbf{r}',t_{\mathrm{ret}}) + \frac{\mathbf{n}}{c\mathcal{R}}\dot{\rho}(\mathbf{r}',t_{\mathrm{ret}}) - \frac{1}{c^2\mathcal{R}}\dot{\mathbf{J}}(\mathbf{r}',t_{\mathrm{ret}})\right). \tag{72}$$

For the static charges and steady currents, the second and the third terms inside the $(\cdots)$ here vanish, while the first term yields the good old Coulomb field of the charge distribution

$\rho(\mathbf{r}')$. But for the time-dependent charges and currents, all three terms inside $(\cdots)$ become important; in particular, the last two terms become more important at the long distances $\mathcal{R} \gg c\tau$ from the charges and currents. And the fact that all charges and currents in eq. (72) are evaluated at the retarded time is also quite important, especially for the radiation by largish antennas. We shall study such radiation in some detail later in class.

Radiation by Point charges

LIÉNARD–WIECHERT POTENTIALS

Thus far, we have focused on the retarded potentials — and hence the EM fields — due to continuous charge distributions $\rho(\mathbf{r},t)$ and $\mathbf{J}(\mathbf{r},t)$. In this section, we turn to the potentials and fields generated by a single charged particle moving along some path $\mathbf{w}(t)$. (I use $\mathbf{w}$ for the particle's location because we have too many different R's in our notations.) For such a moving point particle,

$$\rho(\mathbf{r},t) \;=\; Q\delta^{(3)}(\mathbf{r}-\mathbf{w}(t)), \qquad \mathbf{J}(\mathbf{r},t) \;=\; Q\mathbf{v}(t)\delta^{(3)}(\mathbf{r}-\mathbf{w}(t)) \quad \text{where } \mathbf{v}(t) \stackrel{\text{def}}{=} \frac{d\mathbf{w}(t)}{dt}. \tag{73}$$

The current here follows from the continuity equation: For a moving particle

$$\begin{aligned} \frac{\partial}{\partial t}\big(Q\delta^{(3)}(\mathbf{r}-\mathbf{w}(t))\big) &= Q\frac{\partial \mathbf{w}}{\partial t}\cdot\nabla_{\mathbf{w}}\delta^{(3)}(\mathbf{r}-\mathbf{w}(t)) \\ &= -Q\mathbf{v}(t)\cdot\nabla_{\mathbf{r}}\delta^{(3)}(\mathbf{r}-\mathbf{w}(t)) \\ &= -\nabla_{\mathbf{r}}\cdot\big(Q\mathbf{v}(t)\delta^{(3)}(\mathbf{r}-\mathbf{w}(t))\big), \end{aligned} \tag{74}$$

so we need

$$\nabla\cdot\mathbf{J}(\mathbf{r},t) \;=\; +\nabla_{\mathbf{r}}\cdot\big(Q\mathbf{v}(t)\delta^{(3)}(\mathbf{r}-\mathbf{w}(t))\big) \quad\Longrightarrow\quad \mathbf{J}(\mathbf{r},t) \;=\; Q\mathbf{v}(t)\delta^{(3)}(\mathbf{r}-\mathbf{w}(t)). \tag{75}$$

In the Landau gauge, the EM potentials formally obtain as integrals

$$\begin{aligned} V(\mathbf{r},t) &= \frac{Q}{4\pi\epsilon_0}\iiint d^3\mathrm{Vol}'\,\frac{\delta^{(3)}(\mathbf{r}'-\mathbf{w}(t_{\text{ret}}))}{\mathcal{R}}, \\ \mathbf{A}(\mathbf{r},t) &= \frac{\mu_0 Q}{4\pi}\iiint d^3\mathrm{Vol}'\,\frac{\mathbf{v}(t_{\text{ret}})\delta^{(3)}(\mathbf{r}'-\mathbf{w}(t_{\text{ret}}))}{\mathcal{R}}, \end{aligned} \tag{76}$$

but evaluating these integrals is not as simple as integrating over $\delta^{(3)}(\mathbf{r}'-\text{const})$ because the particle's position $\mathbf{w}(t_{\rm ret})$ is evaluated at the retarded time

$$t_{\rm ret} \;=\; t\;-\;\frac{|\mathbf{r}-\mathbf{r}'|}{c} \tag{77}$$

which also depends on the $\mathbf{r}'$. Consequently,

$$\iiint d^3\mathrm{Vol}'\,\delta^{(3)}(\mathbf{r}'-\mathbf{w}(t_{\rm ret}))\;=\;\frac{1}{\det(\mathcal{J}_{ik})}\quad\text{instead of 1,} \tag{78}$$

where

$$\mathcal{J}_{ik}\;=\;\frac{\partial(r'_k-w_k(t_{\rm ret}))}{\partial r'_i}\,. \tag{79}$$

Specifically,

$$\mathcal{J}_{ik}\;=\;\frac{\partial r'_k}{\partial r'_i}\;-\;\frac{\partial w_k(t_{\rm ret})}{\partial r'_i}\;=\;\delta_{ik}\;-\;\frac{\partial w_k}{\partial t_{\rm ret}}\times\frac{\partial t_{\rm ret}}{\partial r'_i}\;=\;\delta_{ik}\;-\;v_k(t_{\rm ret})\times\frac{n_i}{c} \tag{80}$$

where n_i is the component of the unit vector $\mathbf{n}=(\mathbf{r}-\mathbf{r}')/\mathcal{R}$. The determinant of the matrix (80) is simply

$$\det(\mathcal{J}_{ik})\;=\;1\;-\;\mathbf{v}(t_{\rm ret})\cdot\frac{\mathbf{n}}{c}\,, \tag{81}$$

so

$$\iiint d^3\mathrm{Vol}'\,\delta^{(3)}(\mathbf{r}'-\mathbf{w}(t_{\rm ret}))\;=\;\left[\frac{1}{1\;-\;(\mathbf{v}\cdot\mathbf{n})/c}\right]_{\rm ret} \tag{82}$$

where the subscript 'ret' indicates that both the particle's velocity $\mathbf{v}$ and the unit vector $\mathbf{n}$ from $\mathbf{r}'=\mathbf{w}$ to the observer point $\mathbf{r}$ should be taken at the retarded time $t_{\rm ret}$.

Likewise, evaluating the integrals (76) for the retarded potentials generated by the moving particle, we obtain

$$\begin{aligned} V(\mathbf{r},t)\;&=\;\frac{Q}{4\pi\epsilon_0}\iiint d^3\mathrm{Vol}'\,\frac{\delta^{(3)}(\mathbf{r}'-\mathbf{w}(t_{\rm ret}))}{\mathcal{R}}\\ &=\;\frac{Q}{4\pi\epsilon_0}\left[\frac{1}{\mathcal{R}\,\big(1-\;(\mathbf{v}\cdot\mathbf{n})/c\big)}\right]_{\rm ret}\,, \end{aligned} \tag{83}$$

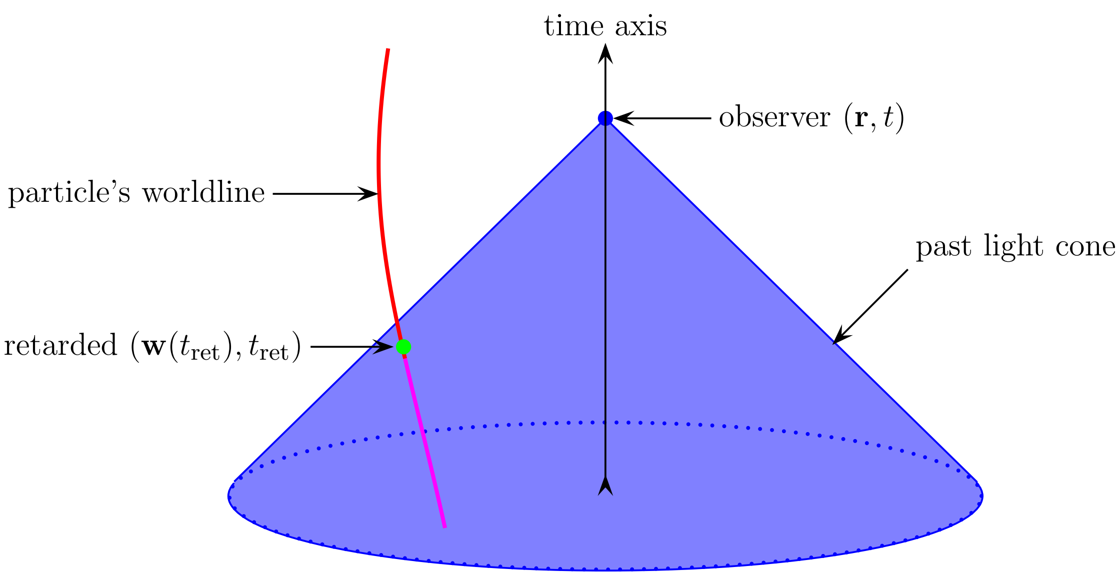
$$\begin{aligned} \mathbf{A}(\mathbf{r},t) &= \frac{\mu_0 Q}{4\pi} \iiint d^3\mathrm{Vol}' \, \frac{\mathbf{v}(t_{\mathrm{ret}})\delta^{(3)}(\mathbf{r}'-\mathbf{w}(t_{\mathrm{ret}}))}{\mathcal{R}} \\ &= \frac{\mu_0 Q}{4\pi} \left[\frac{\mathbf{v}}{\mathcal{R}\left(1-(\mathbf{v}\cdot\mathbf{n})/c\right)} \right]_{\mathrm{ret}} . \end{aligned} \tag{84}$$

Note: in both of these formulae, we have identified $\mathbf{r}'$ with the particle's location $\mathbf{w}(t_{\mathrm{ret}})$, hence

$$\mathcal{R} = |\mathbf{r}-\mathbf{w}(t_{\mathrm{ret}})|\,, \qquad \mathbf{n} = \frac{\mathbf{r}-\mathbf{w}(t_{\mathrm{ret}})}{|\mathbf{r}-\mathbf{w}(t_{\mathrm{ret}})|} . \tag{85}$$

The potentials (83) and (84) are called *Liénard–Wiechert potentials* after Alfred–Marie Liénard and Emil Wiechert who derived them back in 1898–1900 and used them to understand the EM radiation by a moving charge. Although they did it before the Special Relativity was discovered, their results are valid for all particle speeds — slow or relativistic, as long they are slower than light. In particular, the $1/(1-(\mathbf{v}\cdot\mathbf{n})/c)$ — often written as $1/(1-\boldsymbol{\beta}\cdot\mathbf{n})$ for $\boldsymbol{\beta}=\mathbf{v}/c$ — in the Liénard–Wiechert potentials (83) and (84) is very important for the EM radiation by relativistic particles.

The physical origin of the Liénard–Wiechert factor $1/(1-\boldsymbol{\beta}\cdot\mathbf{n})$ is basic spacetime geometry. In spacetime, the particle's motion $\mathbf{w}(t')$ makes a worldline, and for any particular observer time t and location $\mathbf{r}$, the retarded time t_{ret} obtains as an intersection of the particle's worldline with the past light cone of the observer:



Thus, the retarded time $t_{\rm ret}$ is an implicit function of the observer location and time $(\mathbf{r}, t)$ defined by the equation

$$c(t - t_{\rm ret}) \;=\; |\mathbf{r} - \mathbf{w}(t_{\rm ret})|\,. \tag{86}$$

Note that at a fixed observer location $\mathbf{r}$, the retarded time runs at a different rate than the observer time: In light of eq. (86),

$$c\,dt \;-\; c\,dt_{\rm ret} \;=\; d\,|\mathbf{r} - \mathbf{w}(t_{\rm ret})| \;=\; -\mathbf{n}\cdot d\mathbf{w}(t_{\rm ret}) \;=\; -\mathbf{n}\cdot\mathbf{v}\,dt_{\rm ret} \tag{87}$$

and hence

$$\frac{dt_{\rm ret}}{dt} \;=\; \frac{c}{c - \mathbf{n}\cdot\mathbf{v}} \;=\; \frac{1}{1 - \boldsymbol{\beta}\cdot\mathbf{n}}\,. \tag{88}$$

Also, the volume of a moving body appears to the observer to be stretched or shrunk by the same factor. For example, suppose a fast train moves directly towards you along the x axis. Because the train's caboose is further away from your than the locomotive, you see it later than the locomotive, and during this little time difference, the locomotive gets a bit closer to you, which makes the *apparent length* L' of the train longer than its true length L. Indeed, suppose you observe the train from the location $x = 0$, the train's locomotive is at $x_\ell(t) = x_0 - vt$, and the caboose at $x_c(t) = x_l(t) + L = x_0 + L - vt$. Then you observe the locomotive at $x_\ell(t_\ell^{\rm ret})$ for

$$t \;=\; t_\ell^{\rm ret} \;+\; \frac{1}{c}(x_0 - vt_\ell^{\rm ret}) \quad\Longrightarrow\quad t_\ell^{\rm ret} \;=\; \frac{tc - x_0}{c - v} \quad\Longrightarrow\quad x_\ell(t_\ell^{\rm ret}) \;=\; \frac{c}{c - v}\,(x_0 - vt). \tag{89}$$

Likewise, you observe the caboose at $x_c(t_c^{\rm ret})$ for

$$t \;=\; t_c^{\rm ret} + \frac{1}{c}(x_0 + L - vt_c^{\rm ret}) \quad\Longrightarrow\quad t_c^{\rm ret} \;=\; \frac{tc - x_0 - L}{c - v} \quad\Longrightarrow\quad x_c(t_c^{\rm ret}) \;=\; \frac{c}{c - v}\,(x_0 + L - vt). \tag{90}$$

Therefore, the apparent train length is

$$L' \;=\; x_c(t_c^{\rm ret}) \;-\; x_\ell(t_\ell^{\rm ret}) \;=\; \frac{c}{c - v} \times L. \tag{91}$$

On the other hand, the retarded time is the same across any particular cross-section of the train, so its apparent height and width are the same as its real heights and width.

Consequently, the train's volume $\mathcal{V}$ scales the same as the length L:

$$\frac{\text{apparent volume}}{\text{real volume}} = \frac{L'}{L} = \frac{c}{c-v} = \frac{1}{1-\beta}. \tag{92}$$

Now suppose the train moves in the opposite direction, directly away from you. In this case, you see the caboose at an earlier retarded time than the locomotive, and during this time difference, the caboose moves towards the locomotive, thus making the train appear shorter than it is. Repeating the above algebra for this case, we get a few different signs, so we end up with

$$\frac{\text{apparent volume}}{\text{real volume}} = \frac{L'}{L} = \frac{c}{c+v} = \frac{1}{1+\beta}. \tag{93}$$

Finally, let's look at the train from some point away from the tracks, so our line of sight is at angle $\pi-\theta$ to the train's direction. (From the train's point of view, we are located at angle θ from the train's direction, hence $\boldsymbol{\beta}\cdot\mathbf{n}=(v/c)\cos\theta$. Remember that the unit vector $\mathbf{n}$ points from the train towards you rather than from you to the train.) Assuming we are far enough away that θ stays approximately constant during the observation, we find that the train's apparent dimension *in the direction of the line of site* changes by the factor

$$\frac{c}{c-v_{\|}} = \frac{c}{c-v\cos\theta} \tag{94}$$

while the apparent dimensions in the other 2 directions $\perp$ to the line of sight are the same as the true dimensions. Therefore, for the train's volume we end up with

$$\frac{\text{apparent volume}}{\text{real volume}} = \frac{c}{c-v\cos\theta} = \frac{1}{1-\boldsymbol{\beta}\cdot\mathbf{n}}. \tag{95}$$

I have spent a bit of time explaining this geometric factor because it's precisely this factor that affects in the δ-function integral (82). And consequently, it is this geometric factor which enters the *Liénard–Wiechert potentials* (83) and (84).

For future convenience, let me rewrite eqs. (83) and (84) as

$$V(\mathbf{r},t) = \frac{Q}{4\pi\epsilon_0}\frac{1}{F(\mathbf{r},t_{\text{ret}})}, \tag{96}$$

$$\mathbf{A}(\mathbf{r},t) \;=\; \frac{Q\mu_0}{4\pi}\,\frac{\mathbf{v}(t_{\rm ret})}{F(\mathbf{r},t_{\rm ret})} \;=\; \frac{\mathbf{v}(t_{\rm ret})}{c^2} * V(\mathbf{r},t)\,, \tag{97}$$

$$\text{for}\quad F(\mathbf{r},t_{\rm ret}) \;\stackrel{\rm def}{=}\; \left[\mathcal{R} * \left(1 - \frac{\mathbf{v}\cdot\mathbf{n}}{c}\right)\right]_{\rm ret} \;=\; \left[|\mathbf{r}-\mathbf{w}| \,-\, (\mathbf{r}-\mathbf{w})\cdot\frac{\mathbf{v}}{c}\right]_{\rm ret}. \tag{98}$$

Example

As an example of *Liénard–Wiechert potentials*, consider the potentials of a particle moving at constant velocity $\mathbf{v} = \text{const}$. For simplicity, let's choose our coordinate system such that the particle is at the origin at $t = 0$, thus

$$\mathbf{w}(t) \;=\; t\mathbf{v}. \tag{99}$$

An observer located at $\mathbf{r}$ at time t sees this particle at the retarded time such that

$$ct \;-\; ct_{\rm ret} \;=\; |\mathbf{r} - t_{\rm ret}\mathbf{v}|^2\,. \tag{100}$$

Taking the squares of both sided of this equation, we get

$$c^2(t^2 - 2tt_{\rm ret} + t_{\rm ret}^2) \;=\; (\mathbf{r} - t_{\rm ret}\mathbf{v})^2 \;=\; \mathbf{r}^2 \;-\; 2t_{\rm ret}(\mathbf{r}\cdot\mathbf{v}) \;+\; t_{\rm ret}^2\mathbf{v}^2, \tag{101}$$

and hence

$$(c^2 - v^2)t_{\rm ret}^2 \;-\; 2(c^2t - \mathbf{r}\cdot\mathbf{v})t_{\rm ret} \;+\; (c^2t^2 - r^2) \;=\; 0. \tag{102}$$

Solving this quadratic equation, we get

$$t_{\rm ret} \;=\; \frac{c^2t \,+\, (\mathbf{r}\cdot\mathbf{v}) \,\pm\, \sqrt{\mathcal{D}}}{c^2 - v^2} \tag{103}$$

for

$$\mathcal{D} \;=\; (c^2t - \mathbf{r}\cdot\mathbf{v})^2 \;-\; (c^2 - v^2)(c^2t^2 - r^2). \tag{104}$$

To determine the sign in eq. (103), let's take the $v \to 0$ limit in which

$$\mathcal{D} \;\to\; c^2r^2 \quad\Longrightarrow\quad t_{\rm ret} \;\to\; \frac{c^2t \pm cr}{c^2} \;=\; t \;\pm\; \frac{r}{c}\,. \tag{105}$$

Obviously, the right answer for the retarded time is the lower sign in this formula, thus

$$t_{\text{ret}} = \frac{c^2 t - (\mathbf{v}\cdot\mathbf{r}) - \sqrt{\mathcal{D}}}{c^2 - v^2}. \tag{106}$$

Consequently, for a particle moving at constant velocity

$$\begin{aligned}
|\mathbf{r} - \mathbf{w}(t_{\text{ret}})| &= c(t - t_{\text{ret}}) \\
&= \frac{c}{c^2 - v^2}\Big[(c^2 - v^2)t - \Big(c^2 t - (\mathbf{v}\cdot\mathbf{r}) - \sqrt{\mathcal{D}}\Big)\Big] \\
&= \frac{c}{c^2 - v^2}\Big[-v^2 t + (\mathbf{v}\cdot\mathbf{r}) + \sqrt{D}\Big]
\end{aligned} \tag{107}$$

while

$$\begin{aligned}
\mathbf{v}\cdot(\mathbf{r} - \mathbf{w}(t_{\text{ret}})) &= \mathbf{v}\cdot(\mathbf{r} - t_{\text{ret}}\mathbf{v}) = (\mathbf{v}\cdot\mathbf{r}) - v^2 t_{\text{ret}} \\
&= (\mathbf{v}\cdot\mathbf{r}) - \frac{v^2}{c^2 - v^2}\Big[c^2 t - (\mathbf{v}\cdot\mathbf{r}) - \sqrt{\mathcal{D}}\Big] \\
&= \frac{c^2}{c^2 - v^2}\Big((\mathbf{v}\cdot\mathbf{r}) - v^2 t\Big) + \frac{v^2\sqrt{\mathcal{D}}}{c^2 - v^2},
\end{aligned} \tag{108}$$

hence

$$\begin{aligned}
F(\mathbf{r}, t_{\text{ret}}) &= |\mathbf{r} - \mathbf{w}(t_{\text{ret}})| - (\mathbf{r} - \mathbf{w}(t_{\text{ret}}))\cdot\frac{\mathbf{v}}{c} \\
&= \frac{c}{c^2 - v^2}\Big[-v^2 t + (\mathbf{v}\cdot\mathbf{r}) + \sqrt{\mathcal{D}}\Big] - \frac{c}{c^2 - v^2}\Big((\mathbf{v}\cdot\mathbf{r}) - v^2 t\Big) - \frac{v^2\sqrt{\mathcal{D}}}{c(c^2 - v^2)} \\
&= \frac{c\sqrt{\mathcal{D}}}{c^2 - v^2} - \frac{v^2\sqrt{\mathcal{D}}}{c(c^2 - v^2)} = \frac{\sqrt{\mathcal{D}}}{c}.
\end{aligned} \tag{109}$$

According to eqs. (96) and (97), this immediately leads to the Liénard–Wiechert potentials

$$\begin{aligned}
V(\mathbf{r}, t) &= \frac{Q}{4\pi\epsilon_0}\frac{c}{\sqrt{\mathcal{D}}}, \\
\mathbf{A}(\mathbf{r}, t) &= \frac{Q}{4\pi\epsilon_0 c}\frac{\mathbf{v}}{\sqrt{\mathcal{D}}}, \\
\text{where}\quad \mathcal{D} &= (c^2 t - \mathbf{r}\cdot\mathbf{v})^2 + (c^2 - v^2)(r^2 - c^2 t^2).
\end{aligned} \tag{110}$$

To complete this example, let's calculate the electric and magnetic field for the charge moving at constant velocity. In light of eqs.(96), (97), and (110),

$$\begin{aligned}\mathbf{E} &= -\nabla V - \frac{\partial\mathbf{A}}{\partial t} = -\left(\nabla + \frac{\mathbf{v}}{c^2}\frac{\partial}{\partial t}\right)V\\ &= +\frac{Q}{8\pi\epsilon_0}\frac{c}{\mathcal{D}^{3/2}}\left(\nabla + \frac{\mathbf{v}}{c^2}\frac{\partial}{\partial t}\right)\mathcal{D},\end{aligned}\tag{111}$$

$$\begin{aligned}\mathbf{B} &= \nabla\times\left(\mathbf{A} = V * \frac{\mathbf{v}}{c^2}\right) = \nabla V \times \frac{\mathbf{v}}{c^2}\\ &= -\frac{Q}{8\pi\epsilon_0}\frac{1}{c\mathcal{D}^{3/2}}\nabla\mathcal{D}\times\mathbf{v} = +\frac{Q}{8\pi\epsilon_0}\frac{1}{c\mathcal{D}^{3/2}}\mathbf{v}\times\nabla\mathcal{D},\end{aligned}\tag{112}$$

where

$$\nabla\mathcal{D} = 2(\mathbf{r}\cdot\mathbf{v} - ct^2)\mathbf{v} + 2(c^2 - v^2)\mathbf{r},\tag{113}$$

$$\frac{\partial}{\partial t}\mathcal{D} = -2c^2(\mathbf{r}\cdot\mathbf{v} - ct^2)t - 2c^2(c^2 - v^2)t,\tag{114}$$

hence

$$\left(\nabla + \frac{\mathbf{v}}{c^2}\frac{\partial}{\partial t}\right)\mathcal{D} = 0 + 2(c^2 - v^2)(\mathbf{r} - \mathbf{v}t),\tag{115}$$

$$\mathbf{v}\times\nabla\mathcal{D} = 0 + 2(c^2 - v^2)(\mathbf{v}\times\mathbf{r}),\tag{116}$$

and therefore

$$\mathbf{E}(\mathbf{r},t) = \frac{Q}{4\pi\epsilon_0}\frac{c(c^2 - v^2)}{\mathcal{D}^{3/2}}(\mathbf{r} - \mathbf{v}t),\tag{117}$$

$$\mathbf{B}(\mathbf{r},t) = \frac{Q}{4\pi\epsilon_0}\frac{(c^2 - v^2)}{c\mathcal{D}^{3/2}}(\mathbf{v}\times\mathbf{r}) = \frac{\mathbf{v}}{c^2}\times\mathbf{E}(\mathbf{r},t).\tag{118}$$

Note that the electric field points out radially from the *current* location $\mathbf{w}(t) = \mathbf{v}t$ of the charge rather than its retarded location $\mathbf{w}(t_{\rm ret})$!

Finally, let's simplify the formula for the $\mathcal{D}$. Working in the cylindrical coordinate system where the particles moves along the z axis, we have

$$\begin{aligned}\mathcal{D} &= (c^2t - \mathbf{r}\cdot\mathbf{v})^2 + (c^2 - v^2)(r^2 - c^2t^2)\\ &= (c^2t - vz)^2 + (c^2 - v^2)(z^2 + s^2 - c^2t^2)\\ &= t^2\times\left[c^4 - (c^2 - v^2)c^2 = c^2v^2\right] - 2tz\times vc^2\\ &\qquad + z^2\times\left[v^2 + (c^2 - v^2) = c^2\right] + s^2\times(c^2 - v^2)\\ &= c^2(z - vt)^2 + (c^2 - v^2)s^2,\end{aligned}\tag{119}$$

hence

$$\frac{c(c^2-v^2)}{\mathcal{D}^{3/2}} \;=\; \frac{\gamma}{[\gamma^2(z-vt)^2+s^2]^{3/2}} \tag{120}$$

where

$$\gamma \stackrel{\rm def}{=} \sqrt{\frac{c^2}{c^2-v^2}} \tag{121}$$

is the Lorentz contraction factor for lengths of a moving body in the direction of its velocity. In this context, the denominator of eq. (120) is simply the cube of the distance from the moving body before the Lorentz-contraction of the z direction.

Anyway, in light of eq. (120), the electric and the magnetic fields of a uniformly moving point charge are

$$\begin{aligned}
\mathbf{E}(s,z,t) &= \frac{Q}{4\pi\epsilon_0}\,\frac{\gamma\big((z-vt)\hat{\mathbf{z}}+s\hat{\mathbf{s}}\big)}{[\gamma^2(z-vt)^2+s^2]^{3/2}}\,,\\
\mathbf{B}(s,z,t) &= \frac{\mu_0 Q}{4\pi}\,\frac{\gamma v s\hat{\boldsymbol{\phi}}}{[\gamma^2(z-vt)^2+s^2]^{3/2}}\,.
\end{aligned} \tag{122}$$

In the non-relativistic limit of $v \ll c$, we have $\gamma \approx 1$, so the electric field becomes the instantaneous Coulomb field of the moving charge:

$$\mathbf{E} \;\to\; \frac{Q}{4\pi\epsilon_0}\,\frac{(z-vt)\hat{\mathbf{z}}+s\hat{\mathbf{s}}}{[(z-vt)^2+s^2]^{3/2}} \;=\; \frac{Q}{4\pi\epsilon_0}\,\frac{\mathbf{r}-\mathbf{v}t}{|\mathbf{r}-\mathbf{v}t|^3}\,. \tag{123}$$

But for the particles moving at speeds comparable to c, the Lorentzian γ factor becomes important.

Fields of a Moving Charge: General case

Now let's go back from our example to the general case of a charged particle moving at a non-uniform velocity $\mathbf{v}(t) \neq$ const. Thus far, we have found the Liénard–Wiechert potentials

$$V(\mathbf{r},t) \;=\; \frac{Q}{4\pi\epsilon_0}\,\frac{1}{F(\mathbf{r},t_{\rm ret})}\,, \tag{96}$$

$$\mathbf{A}(\mathbf{r},t) \;=\; \frac{\mathbf{v}(t_{\rm ret})}{c^2} * V(\mathbf{r},t)\,, \tag{97}$$

$$\text{for}\quad F(\mathbf{r},t_{\text{ret}}) \stackrel{\text{def}}{=} \left[\mathcal{R}*\left(1-\frac{\mathbf{v}\cdot\mathbf{n}}{c}\right)\right]_{\text{ret}} = \left[|\mathbf{r}-\mathbf{w}| - (\mathbf{r}-\mathbf{w})\cdot\frac{\mathbf{v}}{c}\right]_{\text{ret}} \tag{98}$$

for this general case, so let's use them to calculate the electric and the magnetic fields. At first blush, the EM fields should obtain by a straightforward differentiation,

$$\mathbf{E} = -\nabla V - \frac{\partial\mathbf{A}}{\partial t} \qquad \mathbf{B} = \nabla\times\mathbf{B}. \tag{124}$$

However, the Liénard–Wiechert equations (96) and (97) give the potentials as functions of $\mathbf{r}$ and the retarded time t_{ret} — which in turn depends on both $\mathbf{r}$ and t, — so taking their derivatives WRT $\mathbf{r}$ and t is not so straightforward.

Our first step is to sort the interdependence of these variables. Taking the differential of the basic equation for the retarded time

$$ct - ct_{\text{ret}} = |\mathbf{r}-\mathbf{w}(t_{\text{ret}})|\,, \tag{125}$$

we get

$$cdt - cdt_{\text{ret}} = \mathbf{n}\cdot(d\mathbf{r}-\mathbf{v}dt_{\text{ret}}) \tag{126}$$

and hence

$$dt_{\text{ret}} = \frac{c\,dt - \mathbf{n}\cdot d\mathbf{r}}{c-\mathbf{n}\cdot v}\,. \tag{127}$$

Consequently,

$$\left[\frac{\partial t_{\text{ret}}}{\partial t}\right]_{@\text{fixed}\,\mathbf{r}} = \frac{c}{c-\mathbf{n}\cdot\mathbf{v}}\,, \qquad \left[\nabla t_{\text{ret}}\right]_{@\text{fixed}\,t} = \frac{-\mathbf{n}}{c-\mathbf{n}\cdot\mathbf{v}}\,, \tag{128}$$

and hence for any function $F(\mathbf{r},t_{\text{ret}})$ of the observer location but the retarded time,

$$\begin{aligned}
\left[\frac{\partial F}{\partial t}\right]_{@\text{fixed}\,\mathbf{r}} &= \left[\frac{\partial F}{\partial t_{\text{ret}}}\right]*\left[\frac{\partial t_{\text{ret}}}{\partial t}\right] = \left[\frac{\partial F}{\partial t_{\text{ret}}}\right]_{@\text{fixed}\,\mathbf{r}} * \frac{c}{c-\mathbf{n}\cdot\mathbf{v}}\,,\\
\left[\nabla F\right]_{@\text{fixed}\,t} &= \left[\nabla F\right]_{@\text{fixed}\,t_{\text{ret}}} + \left[\frac{\partial F}{\partial t_{\text{ret}}}\right]_{@\text{fixed}\,\mathbf{r}} * \left[\nabla t_{\text{ret}}\right]_{@\text{fixed}\,t}\\
&= \left[\nabla F\right]_{@\text{fixed}\,t_{\text{ret}}} + \left[\frac{\partial F}{\partial t_{\text{ret}}}\right]_{@\text{fixed}\,\mathbf{r}} * \frac{-\mathbf{n}}{c-\mathbf{n}\cdot\mathbf{v}}\,.
\end{aligned}$$

In particular, for $F(\mathbf{r},t_{\rm ret})$ as in eq. (98),

$$\begin{aligned} \Big[\nabla F\Big]_{@\text{fixed}\,t_{\rm ret}} &= \mathbf{n} - \frac{\mathbf{v}}{c}\,, && (129)\\ \left[\frac{\partial F}{\partial t_{\rm ret}}\right]_{@\text{fixed}\,\mathbf{r}} &= -\mathbf{n}\cdot\mathbf{v} + \frac{v^2}{c} - \frac{\dot{\mathbf{v}}}{c}\cdot(\mathbf{r}-\mathbf{w}), && (130) \end{aligned}$$

and therefore

$$\begin{aligned} \left[\frac{\partial F}{\partial t}\right]_{@\text{fixed}\,\mathbf{r}} &= \frac{1}{c-\mathbf{n}\cdot\mathbf{v}}\Big[-c\mathbf{n}\cdot\mathbf{v} + v^2 - \dot{\mathbf{v}}\cdot(\mathbf{r}-\mathbf{w})\Big], && (131)\\ \Big[\nabla F\Big]_{@\text{fixed}\,t} &= \mathbf{n} - \frac{\mathbf{v}}{c} - \frac{\mathbf{n}}{c(c-\mathbf{n}\cdot\mathbf{v})}\Big[-c\mathbf{n}\cdot\mathbf{v} + v^2 - \dot{\mathbf{v}}\cdot(\mathbf{r}-\mathbf{w})\Big]. && (132) \end{aligned}$$

Using all these formulae, we may finally take derivatives of the Liénard–Wiechert potentials (96) and (97). Starting with the gradient of the scalar potential, we have

$$\begin{aligned} \nabla V &= \frac{Q}{4\pi\epsilon_0}\nabla\frac{1}{F} = -\frac{Q}{4\pi\epsilon_0}\frac{\nabla F}{F^2}\\ &= \frac{Q}{4\pi\epsilon_0}\frac{1}{F^2}\left[-\mathbf{n} + \frac{\mathbf{v}}{c} + \frac{\mathbf{n}}{c-(\mathbf{n}\cdot\mathbf{v})}\left(-(\mathbf{n}\cdot\mathbf{v}) + \frac{v^2}{c} - \dot{\mathbf{v}}\cdot\frac{(\mathbf{r}-\mathbf{w})}{c}\right)\right]\\ &\qquad \langle\!\langle \text{ using } F = \mathcal{R}(1-\mathbf{n}\cdot\boldsymbol{\beta}) \text{ for } \boldsymbol{\beta} = \mathbf{v}/c \,\rangle\!\rangle\\ &= \frac{Q}{4\pi\epsilon_0\,\mathcal{R}^2}\left[\frac{\boldsymbol{\beta}-\mathbf{n}}{(1-\boldsymbol{\beta}\cdot\mathbf{n})^2} + \frac{\mathbf{n}}{(1-\boldsymbol{\beta}\cdot\mathbf{n})^3}\left(\beta^2 - \mathbf{n}\cdot\boldsymbol{\beta} + \frac{\mathcal{R}}{c^2}\mathbf{n}\cdot\dot{\mathbf{v}}\right)\right]\\ &= \frac{Q}{4\pi\epsilon_0}\left[\frac{\boldsymbol{\beta}}{\mathcal{R}^2(1-\boldsymbol{\beta}\cdot\mathbf{n})^2} - \frac{(1-\beta^2)\mathbf{n}}{\mathcal{R}^2(1-\boldsymbol{\beta}\cdot\mathbf{n})^3} + \frac{\mathbf{n}(\mathbf{n}\cdot\dot{\mathbf{v}})}{c^2\mathcal{R}(1-\boldsymbol{\beta}\cdot\mathbf{n})^3}\right]. \end{aligned} \tag{133}$$

Next, the time derivative of the vector potential. Since $\mathbf{A}\propto\mathbf{v}/F$ where both $\mathbf{v}$ and F are at the retarded time, we start with

$$\begin{aligned} \frac{\partial}{\partial t_{\rm ret}}\left(\frac{\mathbf{v}}{F}\right) &= \frac{\dot{\mathbf{v}}}{F} - \frac{\mathbf{v}}{F^2}\frac{\partial F}{\partial t_{\rm ret}}\\ &= \frac{\dot{\mathbf{v}}}{\mathcal{R}(1-\boldsymbol{\beta}\cdot\mathbf{n})} - \frac{c\boldsymbol{\beta}}{\mathcal{R}^2(1-\boldsymbol{\beta}\cdot\mathbf{n})^2}\left(-c\boldsymbol{\beta}\cdot\mathbf{n} + c\beta^2 - \frac{\mathcal{R}}{c}(\mathbf{n}\cdot\dot{\mathbf{v}})\right)\\ &= \frac{\dot{\mathbf{v}}(1-\boldsymbol{\beta}\cdot\mathbf{n}) + \boldsymbol{\beta}(\mathbf{n}\cdot\dot{\mathbf{v}})}{\mathcal{R}(1-\boldsymbol{\beta}\cdot\mathbf{n})^2} + \frac{c^2\boldsymbol{\beta}(\boldsymbol{\beta}\cdot\mathbf{n}-\beta^2)}{\mathcal{R}^2(1-\boldsymbol{\beta}\cdot\mathbf{n})^2}, \end{aligned} \tag{134}$$

hence

$$\frac{\partial}{\partial t}\left(\frac{\mathbf{v}}{F}\right) = \left(\frac{\partial t_{\text{ret}}}{\partial t} = \frac{1}{1-\boldsymbol{\beta}\cdot\mathbf{n}}\right) * \frac{\partial}{\partial t_{\text{ret}}}\frac{\mathbf{v}}{F}$$
$$= \frac{\dot{\mathbf{v}}(1-\boldsymbol{\beta}\cdot\mathbf{n}) + \boldsymbol{\beta}(\mathbf{n}\cdot\dot{\mathbf{v}})}{\mathcal{R}(1-\boldsymbol{\beta}\cdot\mathbf{n})^3} + \frac{c^2\boldsymbol{\beta}(\boldsymbol{\beta}\cdot\mathbf{n} - \beta^2)}{\mathcal{R}^2(1-\boldsymbol{\beta}\cdot\mathbf{n})^3}, \tag{135}$$

and therefore

$$\frac{\partial\mathbf{A}}{\partial t} = \frac{Q}{4\pi\epsilon_0 c^2}\frac{\partial}{\partial t}\left(\frac{\mathbf{v}}{F}\right)$$
$$= \frac{Q}{4\pi\epsilon_0}\frac{\boldsymbol{\beta}(\boldsymbol{\beta}\cdot\mathbf{n} - \beta^2)}{\mathcal{R}^2(1-\boldsymbol{\beta}\cdot\mathbf{n})^3} + \frac{Q}{4\pi\epsilon_0}\frac{\dot{\mathbf{v}}(1-\boldsymbol{\beta}\cdot\mathbf{n}) + \boldsymbol{\beta}(\mathbf{n}\cdot\dot{\mathbf{v}})}{c^2\mathcal{R}(1-\boldsymbol{\beta}\cdot\mathbf{n})^3}. \tag{136}$$

Finally, combining the ∇V and $\dot{\mathbf{A}}$ terms into the electric field, we arrive at

$$\mathbf{E} = -\nabla V - \frac{\partial\mathbf{A}}{\partial t}$$
$$= -\frac{Q}{4\pi\epsilon_0}\left[\frac{\boldsymbol{\beta}}{\mathcal{R}^2(1-\boldsymbol{\beta}\cdot\mathbf{n})^2} - \frac{(1-\beta^2)\mathbf{n}}{\mathcal{R}^2(1-\boldsymbol{\beta}\cdot\mathbf{n})^3} + \frac{\mathbf{n}(\mathbf{n}\cdot\dot{\mathbf{v}})}{c\mathcal{R}(1-\boldsymbol{\beta}\cdot\mathbf{n})^3}\right]$$
$$- \frac{Q}{4\pi\epsilon_0}\left[\frac{\boldsymbol{\beta}(\boldsymbol{\beta}\cdot\mathbf{n} - \beta^2)}{\mathcal{R}^2(1-\boldsymbol{\beta}\cdot\mathbf{n})^3} + \frac{\dot{\mathbf{v}}(1-\boldsymbol{\beta}\cdot\mathbf{n}) + \boldsymbol{\beta}(\mathbf{n}\cdot\dot{\mathbf{v}})}{c^2\mathcal{R}(1-\boldsymbol{\beta}\cdot\mathbf{n})^3}\right]$$
$$\langle\!\langle \text{ after a bit of algebra } \rangle\!\rangle$$
$$= \frac{Q}{4\pi\epsilon_0\,\mathcal{R}^2}\frac{(1-\beta^2)(\mathbf{n}-\boldsymbol{\beta})}{(1-\boldsymbol{\beta}\cdot\mathbf{n})^3} + \frac{Q}{4\pi\epsilon_0\,c^2\mathcal{R}}\frac{\mathbf{n}\times((\boldsymbol{\beta}-\mathbf{n})\times\dot{\mathbf{v}})}{(1-\boldsymbol{\beta}\cdot\mathbf{n})^3}.$$

Or rather,

$$\mathbf{E} = \frac{Q}{4\pi\epsilon_0}\left[\frac{(1-\beta^2)(\mathbf{n}-\boldsymbol{\beta})}{\mathcal{R}^2(1-\boldsymbol{\beta}\cdot\mathbf{n})^3}\right]_{\text{ret}} + \frac{Q\mu_0}{4\pi}\left[\frac{\mathbf{n}\times((\boldsymbol{\beta}-\mathbf{n})\times\dot{\mathbf{v}})}{\mathcal{R}(1-\boldsymbol{\beta}\cdot\mathbf{n})^3}\right]_{\text{ret}}. \tag{137}$$

The green term here is called the *generalized Coulomb field* because for a slowly moving particle it becomes the ordinary Coulomb field; indeed, for $\beta \ll 1$ (*i.e.*, $v \ll c$) the expression inside square brackets becomes simply $\mathbf{n}/\mathcal{R}^2$. Like the ordinary Coulomb field, the generalized Coulomb field scales with the distance as $1/\mathcal{R}^2$ and does not depend on the particle's acceleration $\mathbf{a} = \dot{\mathbf{v}}$. However, it does depend on the particle's velocity due to relativistic effects. In eq. (137), we wrote this generalized Coulomb field in terms of the particle's position and velocity at the retarded time, because that the only way we can write it for a general particle's trajectory. For a known and simple trajectory, we might be able to rewrite this field in terms of the observer time, just as we did in our example of a particle moving at constant velocity. Indeed, for $\mathbf{v} = \text{const}$ one can show that

$$\frac{Q}{4\pi\epsilon_0}\left[\frac{(1-\beta^2)(\mathbf{n}-\boldsymbol{\beta})}{\mathcal{R}^2(1-\boldsymbol{\beta}\cdot\mathbf{n})^3}\right]_{\text{ret}} = \frac{Q}{4\pi\epsilon_0}\left[\frac{\gamma(\mathbf{r}-\mathbf{v}t)}{[\gamma^2(z-vt)^2+s^2]^{3/2}}\right]_{\text{time}\,t}^{\text{observer}}, \tag{138}$$

so the generalized Coulomb field is exactly as we had in our example. But let me skip the proof of this formula, as we have already had too many painful calculations.

The red term in the electric field (137) is called the *acceleration field* because it depends on the particle's acceleration $\mathbf{a} = \dot{\mathbf{v}}$. Unlike the generalized Coulomb field, the acceleration field scales with distance as $1/\mathcal{R}$ rather than $1/\mathcal{R}^2$, so for $\dot{\mathbf{v}} \neq 0$ this term dominates the electric field at long distances. It is this term that's responsible for the *EM radiation* by an accelerating charged particle, so its also called the *radiation field*. We shall explore this radiation in the next section of these notes.

But first, let's calculate the magnetic field $\mathbf{B} = \nabla\times\mathbf{A}$ that accompanies the electric field (137). Since $\mathbf{A} \propto \mathbf{v}/F$ where both $\mathbf{v}$ and F are taken at the retarded time, we have

$$\left(\nabla\times\frac{\mathbf{v}}{F}\right)_{@\text{fixed}\,t} = \left(\nabla\times\frac{\mathbf{v}}{F}\right)_{@\text{fixed}\,t_{\text{ret}}} + (\nabla t_{\text{ret}})_{@\text{fixed}\,t}\times\frac{\partial}{\partial t_{\text{ret}}}\left(\frac{\mathbf{v}}{F}\right), \tag{139}$$

where

$$\begin{aligned}\left(\nabla\times\frac{\mathbf{v}}{F}\right)_{@\text{fixed}\,t_{\text{ret}}} &= \frac{\mathbf{v}}{F^2}\times\Big(\nabla F\Big)_{@\text{fixed}\,t_{\text{ret}}}\\ &= \frac{\mathbf{v}}{\mathcal{R}^2(1-\boldsymbol{\beta}\cdot\mathbf{n})^2}\times(\mathbf{n}-\boldsymbol{\beta})\\ &= \frac{c\boldsymbol{\beta}\times\mathbf{n}}{\mathcal{R}^2(1-\boldsymbol{\beta}\cdot\mathbf{n})^2}, \end{aligned}\tag{140}$$

$$(\nabla t_{\text{ret}})_{@\text{fixed}\,t} = -\frac{1}{c}\,\frac{\mathbf{n}}{1-\boldsymbol{\beta}\cdot\mathbf{n}}, \tag{141}$$

$$
\begin{aligned}
\frac{\partial}{\partial t_{\text{ret}}}\left(\frac{\mathbf{v}}{F}\right) &= \frac{\dot{\mathbf{v}}}{F} - \frac{\mathbf{v}}{F^2}\frac{\partial F}{\partial t_{\text{ret}}} \\
&= \frac{\dot{\mathbf{v}}}{\mathcal{R}(1-\boldsymbol{\beta}\cdot\mathbf{n})} - \frac{c\boldsymbol{\beta}}{\mathcal{R}^2(1-\boldsymbol{\beta}\cdot\mathbf{n})^2}\left(-c\boldsymbol{\beta}\cdot\mathbf{n} + c\beta^2 - \frac{\mathcal{R}}{c}\mathbf{n}\cdot\dot{\mathbf{v}}\right) \\
&= \frac{\dot{\mathbf{v}} - \dot{\mathbf{v}}(\boldsymbol{\beta}\cdot\mathbf{n}) + \boldsymbol{\beta}(\mathbf{n}\cdot\dot{\mathbf{v}})}{\mathcal{R}(1-\boldsymbol{\beta}\cdot\mathbf{n})^2} + \frac{c^2\boldsymbol{\beta}(\boldsymbol{\beta}\cdot\mathbf{n}-\beta^2)}{\mathcal{R}^2(1-\boldsymbol{\beta}\cdot\mathbf{n})^2}.
\end{aligned}
\tag{142}
$$

Consequently,

$$
\begin{aligned}
\left(\nabla\times\frac{\mathbf{v}}{F}\right)_{@\text{fixed}\,t} &= \frac{c\boldsymbol{\beta}\times\mathbf{n}}{\mathcal{R}^2(1-\boldsymbol{\beta}\cdot\mathbf{n})^2} - \frac{\mathbf{n}}{c(1-\boldsymbol{\beta}\cdot\mathbf{n})}\times\frac{\dot{\mathbf{v}} - \dot{\mathbf{v}}(\boldsymbol{\beta}\cdot\mathbf{n}) + \boldsymbol{\beta}(\mathbf{n}\cdot\dot{\mathbf{v}})}{\mathcal{R}(1-\boldsymbol{\beta}\cdot\mathbf{n})^2} \\
&\quad - \frac{\mathbf{n}}{c(1-\boldsymbol{\beta}\cdot\mathbf{n})}\times\frac{c^2\boldsymbol{\beta}(\boldsymbol{\beta}\cdot\mathbf{n}-\beta^2)}{\mathcal{R}^2(1-\boldsymbol{\beta}\cdot\mathbf{n})^2} \\
&\quad \langle\!\langle \text{ after a bit of algebra } \rangle\!\rangle \\
&= \frac{c(1-\beta^2)(\boldsymbol{\beta}\times\mathbf{n})}{\mathcal{R}^2(1-\boldsymbol{\beta}\cdot\mathbf{n})^3} + \frac{\mathbf{n}\times(\mathbf{n}\times((\boldsymbol{\beta}-n)\times\dot{\mathbf{v}}))}{c\mathcal{R}(1-\boldsymbol{\beta}\cdot\mathbf{n})^3}
\end{aligned}
\tag{143}
$$

and therefore

$$
\mathbf{B} = \frac{Q}{4\pi\epsilon_0 c}\left[\frac{(1-\beta^2)(\boldsymbol{\beta}\times\mathbf{n})}{\mathcal{R}^2(1-\boldsymbol{\beta}\cdot\mathbf{n})^3}\right]_{\text{ret}} + \frac{Q\mu_0}{4\pi c}\left[\frac{\mathbf{n}\times(\mathbf{n}\times((\boldsymbol{\beta}-\mathbf{n})\times\dot{\mathbf{v}}))}{\mathcal{R}(1-\boldsymbol{\beta}\cdot\mathbf{n})^3}\right]_{\text{ret}}. \tag{144}
$$

Comparing this formula to eq. (137) for the electric field, we immediately see that

$$
\mathbf{B}(\mathbf{r},t) = \frac{\mathbf{n}_{\text{ret}}}{c}\times\mathbf{E}(\mathbf{r},t), \tag{145}
$$

including both the generalized Coulomb and the acceleration terms in the electric field.

Radiation by an Accelerated Charge

Consider the energy flow of the EM fields (137) and (144) generated by the moving point charge. In light of eq. (145) related the electric and the magnetic fields to each other, the Poynting vector is

$$
\mathbf{S} = \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B} = \frac{1}{\mu_0 c}\mathbf{E}\times(\mathbf{n}\times\mathbf{E}) = \frac{1}{Z_0}\big(E^2\mathbf{n} - (\mathbf{n}\cdot\mathbf{E})\mathbf{E}\big). \tag{146}
$$

Some of this energy flow is simply the electrostatic energy that moves along with the charge, some of it swirls around the charge in a more complicated fashion, and some of it flows away from the charge all the way to infinity. The *electromagnetic radiation* is the last part: the energy which flows all the way to infinity. To sort the EM radiation from the other kind of EM energy flows, let's surround the charge — or rather its retarded position $\mathbf{w}(t_{\rm ret})$ — by a sphere of a large radius R, so at a delayed time $t = t_{\rm ret} + R/c$ we can measure the EM power emitted by the charge at the time $t_{\rm ret}$. Looking at the EM power flowing in a particular direction, we have

$$\frac{\text{power}}{\text{solid angle}} \;=\; \frac{dP}{d\Omega} \;=\; \frac{d\,\mathbf{r}area}{d\Omega}\cdot\mathbf{S} \;=\; R^2\mathbf{n}\cdot\mathbf{S} \tag{147}$$

so the EM radiation is the part of this power which stays finite in the $R\to\infty$ limit.

Now look at the scaling of the electric field

$$\mathbf{E} \;=\; \frac{Q}{4\pi\epsilon_0}\left[\frac{(1-\beta^2)(\mathbf{n}-\boldsymbol{\beta})}{\mathcal{R}^2(1-\boldsymbol{\beta}\cdot\mathbf{n})^3}\right]_{\rm ret} \;+\; \frac{Q\mu_0}{4\pi}\left[\frac{\mathbf{n}\times((\boldsymbol{\beta}-\mathbf{n})\times\dot{\mathbf{v}})}{\mathcal{R}(1-\boldsymbol{\beta}\cdot\mathbf{n})^3}\right]_{\rm ret} \tag{137}$$

with the distance $\mathcal{R} = R$ from the charge: The generalized Coulomb term scales like $1/R^2$ while the acceleration term scales like $1/R$. Consequently, the Poynting vector (146) — which is proportional to the $\mathbf{E}^2$ — has 3 kinds of terms:

- Coulomb-only $O(E^2_{\rm coulomb})$ terms which scale as $1/R^4$;
- interference $O(E_{\rm Coulomb}\times E_{\rm accel})$ terms which scale as $1/R^3$;
- acceleration-only $O(E^2_{\rm accel})$ terms which scale as $1/R^2$.

In light of eq. (147), *only the acceleration terms constitute the EM radiation* whose power (per solid angle of direction) does not diminish with distance. The other terms corresponds to the other kinds of EM energy flows. Specifically, the Coulomb-only terms describe the electrostatic energy moving along with the charge, while the interference terms correspond to more complicated energy flows around the charge that don't quite reach all the way to infinity.

The bottom line is, the EM radiation stems only from the acceleration term — AKA the radiation term, — in the charge's electric field,

$$\mathbf{E}_{\text{rad}} = \frac{Q\mu_0}{4\pi}\left[\frac{\mathbf{n}\times((\boldsymbol{\beta}-\mathbf{n})\times\mathbf{a})}{\mathcal{R}(1-\boldsymbol{\beta}\cdot\mathbf{n})^3}\right]_{\text{ret}}, . \tag{148}$$

Specifically,

$$\mathbf{S}_{\text{rad}} = \frac{E_{\text{rad}}^2}{Z_0}\,\mathbf{n} \tag{149}$$

since $\mathbf{E}_{\text{rad}}$ is $\perp$ $\mathbf{n}$, and hence

$$\frac{dP_{\text{rad}}}{d\Omega} = \frac{R^2 E_{\text{rad}}^2}{Z_0}\,. \tag{150}$$

Non-relativistic limit

For charges moving with relativistic speeds $v \sim c$, eq. (148) for the radiation field looks rather complicated, so let's start with a much simpler limit of non-relativistic speeds $v \ll c$. In this limit, we may neglect all terms involving $\boldsymbol{\beta} = \mathbf{v}/c$ in eq. (148), which leaves us with a much simpler formula

$$\mathbf{E}_{\text{rad}} = \frac{Q\mu_0}{4\pi\mathcal{R}}\big[-\mathbf{n}\times(\mathbf{n}\times\mathbf{a}) = \mathbf{a} - (\mathbf{n}\cdot\mathbf{a})\mathbf{n}\big]. \tag{151}$$

Consequently

$$\mathbf{E}_{\text{rad}}^2 = \left(\frac{Q\mu_0}{4\pi\mathcal{R}}\right)^2 \mathbf{a}^2\sin^2\theta \tag{152}$$

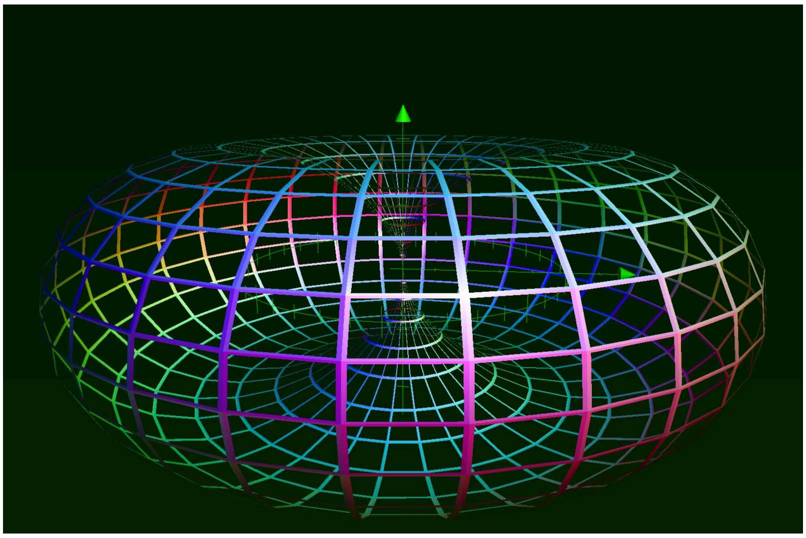
where θ is the angle between the charge's acceleration $\mathbf{a}$ and the direction $\mathbf{n}$ from the charge towards the observer, and therefore

$$\frac{dP_{\text{rad}}}{d\Omega} = \frac{Q^2\mu_0^2}{16\pi^2 Z_0}\,\mathbf{a}^2\sin^2\theta. \tag{153}$$

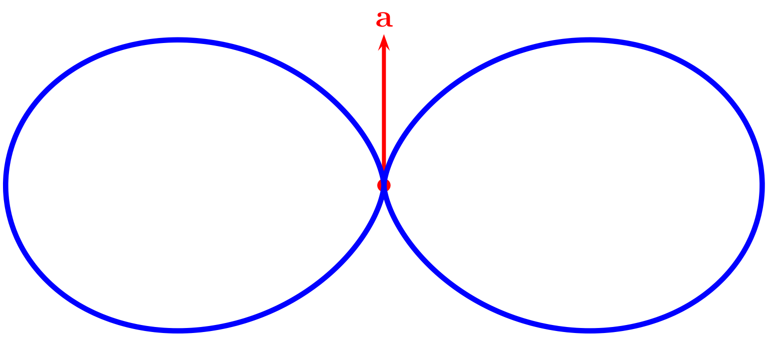
Or using $\mu_0/Z_0 = 1/c$,

$$\frac{dP_{\text{rad}}}{d\Omega} = \frac{Q^2\mu_0\mathbf{a}^2}{16\pi^2 c}\,\sin^2\theta. \tag{154}$$

Note the angular distribution of this radiated power: No radiation is emitted in the direction of the acceleration $\mathbf{a}$ or in the opposite direction; instead, most power is emitted in the directions $\perp$ $\mathbf{a}$, and less power at some sharper (or duller) angles to $\mathbf{a}$. Here is the 3D *radiation power diagram* of the charge accelerating vertically up



and here is its vertical cross-section



Finally, the net power radiated by the charge obtains by integrating the directional power (154) over 4π directions,

$$P_{\rm rad}^{\rm net} \;=\; \oiint d\Omega\,\frac{dP_{\rm rad}}{d\Omega} \;=\; \oiint d\Omega\frac{Q^2\mu_0\mathbf{a}^2}{16\pi^2c}\,\sin^2\theta\,. \tag{155}$$

Using

$$\oiint d\Omega\,\sin^2\theta \;=\; \int\limits_0^\pi d\theta\,\sin\theta\int\limits_0^{2\pi} d\phi, \sin^2\theta \;=\; 2\pi\int\limits_0^\pi \sin^3\theta\,d\theta \;=\; 2\pi\times\tfrac{4}{3}, \tag{156}$$

we arrive at the *Larmor formula* for the net radiated EM power by a non-relativistic accelerating charge:

$$P_{\rm rad}^{\rm net} = \frac{Q^2\mu_0\mathbf{a}^2}{6\pi c}. \tag{157}$$

Example: An electron in a CRT is accelerated by the 2550 V voltage to speed $v = 0.1\,c = 3\cdot 10^7$ m/s slams into the anode and comes to stop over the distance $d = 1$ Å. During the stopping time $t_{\rm stop} = 2d/v = 6.7\cdot 10^{-18}$ s, the electron has rather large acceleration

$$a = \frac{v^2}{2d} = 4.5\cdot 10^{+24}\ \mathrm{m/s^2}, \tag{158}$$

so it radiates EM power

$$P = \frac{(1.6\cdot 10^{-19}\ \mathrm{C})^2\times(4\pi\cdot 10^{-7}\ \mathrm{H/m})\times(4.5\cdot 10^{24}\ \mathrm{m/s^2})^2}{6\pi(3\cdot 10^8\ \mathrm{m/s})} = 1.15\cdot 10^{-4}\ \mathrm{W}. \tag{159}$$

This is a huge power for a single electron! Unfortunately, it's only radiated over the very short stopping time $t_{\rm stop} = 6.7\cdot 10^{-18}$ s, so the net EM energy radiated by the electron is only $U_{\rm rad} = P\times t_{\rm stop} = 7.7\cdot 10^{-23}\ J$, about $1.9\cdot 10^{-7}$ of the kinetic energy it had before stopping.

Another example: *Rutherford atom.*
In the Rutherford's planetary model of an atom, the electrons orbit the nucleus like the planets orbit the Sun, with the Coulomb force playing the role of the gravitational force. Unfortunately, the orbital motion involve continuous acceleration, which makes the electrons radiate EM waves and lose their energy. Eventually, the electrons lose so much energy they fall onto the nucleus in a rather short time. This makes the classical Rutherford model unworkable, so Niels Bohr had to invent Quantum Mechanics (or rather, the early version of quantum mechanics) to solve this problem.

For simplicity, consider the model of a hydrogen atom where a single electron moves around the proton in a circular orbit of radius r. The orbital speed of the electron follows from the Coulomb force as

$$m_e a \;=\; \frac{m_e v^2}{r} \;=\; F_{\rm Coulomb} \;=\; \frac{e^2}{4\pi\epsilon_0\, r^2}\,, \tag{160}$$

hence

$$v^2 \;=\; \frac{e^2}{4\pi\epsilon_0 m_e}\times\frac{1}{r} \tag{161}$$

and the orbital acceleration

$$a \;=\; \frac{v^2}{r} \;=\; \frac{e^2}{4\pi\epsilon_0 m_e}\times\frac{1}{r^2}\,. \tag{162}$$

According to the Larmor formula (157), this makes the electron radiate EM power

$$P \;=\; \frac{e^2\mu_0 a^2}{6\pi c} \;=\; \frac{e^6\mu_0}{96\pi^3\epsilon_0^2 m_e^2 c}\times\frac{1}{r^4} \;=\; \frac{e^6}{96\pi^3\epsilon_0^2 m_e^2 c^3}\times\frac{1}{r^4}\,. \tag{163}$$

But this power comes at the expense of the atom's binding energy

$$U \;=\; +\frac{mv^2}{2} \;-\; \frac{e^2}{4\pi\epsilon_0}\times\frac{1}{r} \;=\; -\frac{e^2}{8\pi\epsilon_0}\times\frac{1}{r}\,, \tag{164}$$

$$P \;=\; -\frac{dU}{dt}\,. \tag{165}$$

Thus, the binding energy decreases with time — *i.e.*, becomes more negative, — which makes the orbital radius r shrink with time. Specifically,

$$-\frac{dU}{dt} \;=\; -\frac{e^2}{8\pi\epsilon_0}\times\frac{1}{r^2}\,\frac{dr}{dt}\,, \tag{166}$$

so equating this energy loss with the EM radiation power emitted by the electron, we arrive at

$$-\frac{e^2}{8\pi\epsilon_0}\times\frac{1}{r^2}\,\frac{dr}{dt} \;=\; \frac{e^6}{96\pi^3\epsilon_0^2 m_e^2 c^3}\times\frac{1}{r^4} \tag{167}$$

and hence

$$3r^2\,\frac{dr}{dt} \;=\; -\frac{e^4}{4\pi^2\epsilon_0^2 m_e^2 c^3}\,. \tag{168}$$

Solving this differential equation for the shrinking radius $r(t)$ is trivial: On the LHS of eq. (168) we have $d(r^3)/dt$ while the RHS is a constant, hence

$$r^3(t) \;=\; r_0^3 \;-\; \frac{e^4}{4\pi^2\epsilon_0^2 m_e^2 c^3} \times t \tag{169}$$

where r_0 is the initial radius of the electron's orbit. Note that according to this equation, the orbital radius shrinks to zero — *i.e.*, the electron falls down onto the nucleus, — in a finite time

$$t_{\text{collapse}} \;=\; r_0^3 \times \frac{4\pi^2\epsilon_0^2 c^3 m_e^2}{e^4}\,. \tag{170}$$

Numerically, for $r_0 = 0.53$ Å (Bohr radius of the hydrogen atom), $t_{\text{collapse}} \approx 1.6 \cdot 10^{-11}$ s, a rather short time.

Relativistic Speeds

For the charges moving at speeds comparable to the speed of light, we have more complicated formulae for the EM power they radiate. Starting with the radiation term

$$\mathbf{E}_{\text{rad}} \;=\; \frac{Q\mu_0}{4\pi}\left[\frac{\mathbf{n}\times((\boldsymbol{\beta}-\mathbf{n})\times\mathbf{a})}{\mathcal{R}(1-\boldsymbol{\beta}\cdot\mathbf{n})^3}\right]_{\text{ret}} \tag{148}$$

in the electric field of the moving point charge, we have $\mathbf{S}_{\text{rad}} = (\mathbf{E}^2_{\text{rad}})\mathbf{n}$ and hence

$$\frac{dP_{\text{obs}}}{d\Omega} \;=\; \frac{R^2\mathbf{E}^2_{\text{rad}}}{Z_0} \;=\; \frac{Q^2\mu_0}{16\pi^2 c}\,\frac{|\mathbf{n}\times((\boldsymbol{\beta}-\mathbf{n})\times\mathbf{a})|^2}{(1-\boldsymbol{\beta}\cdot\mathbf{n})^6}\Bigg|_{\text{ret}}\,. \tag{171}$$

Note: P_{obs} in this formula is the rate at which the radiated energy crosses the distant sphere of radius R in unit of the observer time t. But the energy arrives at the observer at the different rate than it was emitted by the moving particle. Indeed, as we saw earlier in these notes, for a fast-moving particle, the retarded time runs at a different rate than the observer time,

$$\frac{dt_{\text{ret}}}{dt} \;=\; \frac{1}{1-\boldsymbol{\beta}\cdot\mathbf{n}}, \tag{172}$$

so from the moving particle's point of view, the energy it emits per unit of its own time is different from the energy received by the observer per unit of its own time by the inverse factor,

$$\frac{dW}{dt_{\text{ret}}} = (1 - \boldsymbol{\beta} \cdot \mathbf{n}) * \frac{dW}{dt}. \tag{173}$$

Or in terms of the power P_{rad} emitted by the particle by its own measurement of time, — or rather the power $dP_{\text{rad}}/d\Omega$ emitted in a particular direction, — is related to the power received by the observer in that direction as

$$\frac{dP_{\text{rad}}}{d\Omega} = (1 - \boldsymbol{\beta} \cdot \mathbf{n}) * \frac{dP_{\text{obs}}}{d\Omega}. \tag{174}$$

Consequently, eq. (171) for the observed power becomes a slightly different formula

$$\frac{dP_{\text{rad}}}{d\Omega} = \frac{Q^2 \mu_0}{16\pi^2 c} * \frac{|\mathbf{n} \times ((\boldsymbol{\beta} - \mathbf{n}) \times \mathbf{a})|^2}{(1 - \boldsymbol{\beta} \cdot \mathbf{n})^5} \tag{175}$$

for the emitted power.

The angular distribution (175) of the radiation power emitted by a relativistic particle is quite complicated, as it depends on the relative directions of the velocity $\mathbf{v} = \boldsymbol{\beta} c$ and the acceleration $\mathbf{a}$. So for simplicity's sake, let me focus on two special cases, $\mathbf{v} \parallel \mathbf{a}$ and $\mathbf{v} \perp \mathbf{a}$.

1. Particle moving in a straight line but at variable speed, thus $\mathbf{a} \parallel \mathbf{A}$. Consequently, in the numerator of eq. (175)

$$\begin{aligned} \mathbf{n} \times ((\boldsymbol{\beta} - \mathbf{n}) \times \mathbf{a}) &= -\mathbf{n} \times (\mathbf{n} \times \mathbf{a}) \qquad \langle\!\langle \text{ because } \boldsymbol{\beta} \times \mathbf{a} = 0 \rangle\!\rangle \\ &= \mathbf{a} - (\mathbf{n} \cdot \mathbf{a})\mathbf{n} \end{aligned} \tag{176}$$

and hence

$$|\mathbf{n} \times ((\boldsymbol{\beta} - \mathbf{n}) \times \mathbf{a})|^2 = \mathbf{a}^2 - (\mathbf{n} \cdot \mathbf{a})^2 = \mathbf{a^2} \sin^2 \theta \tag{177}$$

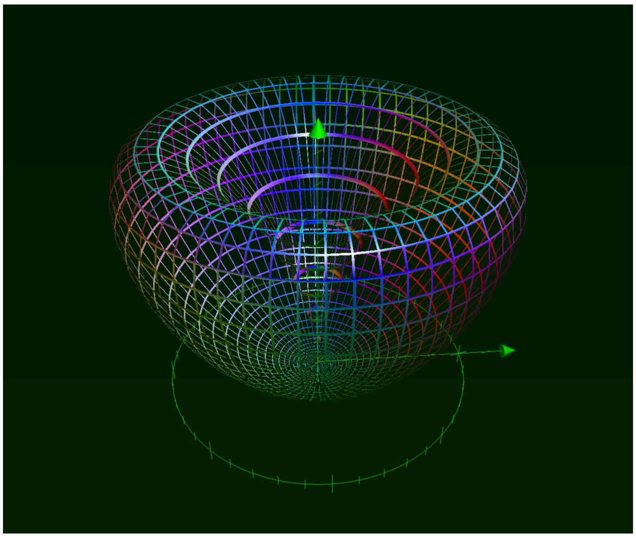
where θ is the angle between the direction $\mathbf{n}$ towards the observer and the direction of the acceleration $\mathbf{a}$ and also the velocity $\mathbf{v}$. At the same time, in the denominator of eq. (175)

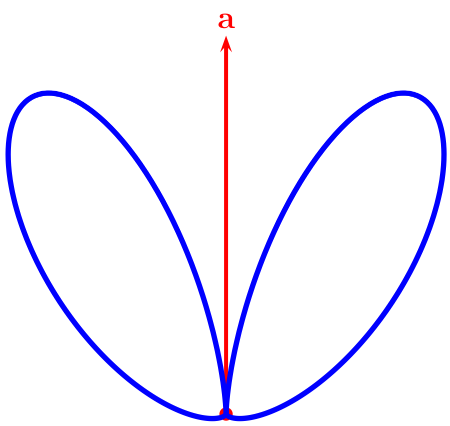
$$(1-\boldsymbol{\beta}\cdot\mathbf{n})^5 \;=\; (1-\beta\cos\theta)^5, \tag{178}$$

thus altogether

$$\frac{dP_{\rm rad}}{d\Omega} \;=\; \frac{Q^2\mu_0\mathbf{a}^2}{16\pi^2c}*\frac{\sin^2\theta}{(1-\beta\cos\theta)^5}\,. \tag{179}$$

Here is the 3D plot of this angular distribution for $\beta=0.6$ and its vertical cross-section





Note that the peak of this angular distribution is shifted forward from $\theta=90^\circ$ to $\theta=32^\circ$, although there is no radiation going directly forward. And at higher speeds approaching the speed of light, the forward shift of the angular distribution becomes stronger and stronger until almost all radiation is emitted in the narrow forward cone $0<\theta<O(1/\gamma)$ (where $\gamma=1/\sqrt{1-\beta^2}$), with the maximal power going in the directions $\theta=1/2\gamma$; but again, no power is emitted directly forward, at $\theta=0$. For the ultra-relativistic particles with $\gamma\gg1$, we may approximate the angular distribution (179) as

$$\frac{dP_{\rm rad}}{d\Omega} \;=\; \frac{Q^2\mu_0\mathbf{a}^2}{16\pi^2c}*\frac{32\gamma^{10}\theta^2}{[1+(\gamma\theta)^2]^5}\,. \tag{180}$$

Now let's find the net power emitted by the particle. Integrating eq. (179) over the directions, we have

$$P_{\rm rad}^{\rm net} \;=\; \oiint d\Omega\,\frac{dP_{\rm rad}}{d\Omega} \;=\; \frac{Q^2\mu_0\mathbf{a}^2}{16\pi^2 c}\oiint d\Omega\,\frac{\sin^2\theta}{(1-\beta\cos\theta)^5}\,, \tag{181}$$

where

$$\oiint d\Omega\,\frac{\sin^2\theta}{(1-\beta\cos\theta)^5} \;=\; \int_0^\pi d\theta\,\sin\theta\int_0^{2\pi}d\phi\,\frac{\sin^2\theta}{(1-\beta\cos\theta)^5} \;=\; 2\pi\int_0^\pi\frac{\sin^3\theta\,d\theta}{(1-\beta\cos\theta)^5}\,. \tag{182}$$

To evaluate this last integral, we change variables from θ to $\xi = 1-\beta\cos\theta$. Consequently,

$$\sin\theta\,d\theta \;=\; -d\cos\theta \;=\; +\frac{d\xi}{\beta} \tag{183}$$

while

$$\sin^2\theta \;=\; 1\;-\;\cos^2\theta \;=\; 1\;-\;\frac{(1-\xi)^2}{\beta^2} \;=\; \frac{-\xi^2+2\xi-(1-\beta^2)}{\beta^2}\,.$$

so the integral (182) becomes

$$\begin{aligned} 2\pi\int_0^\pi\frac{\sin^3\theta\,d\theta}{(1-\beta\cos\theta)^5} &= \frac{2\pi}{\beta^3}\int_{1-\beta}^{1+\beta}d\xi\,\frac{-\xi^2+2\xi-(1-\beta^2)}{\xi^5}\\ &= \frac{2\pi}{\beta^3}\left[\frac{1}{2\xi^2}\;-\;\frac{2}{3\xi^3}\;+\;\frac{(1-\beta^2)}{4\xi^4}\right]\Bigg|_{\xi=1-\beta}^{\xi=1+\beta} \end{aligned} \tag{184}$$

where

$$\frac{1}{2\xi^2}\Bigg|_{\xi=1-\beta}^{\xi=1+\beta} \;=\; \frac{1}{2(1+\beta)^2}\;-\;\frac{1}{2(1-\beta)^2} \;=\; \frac{-2\beta}{(1-\beta^2)^2}\,, \tag{185}$$

$$\frac{-2}{3\xi^2}\Bigg|_{\xi=1-\beta}^{\xi=1+\beta} \;=\; -\frac{2}{3(1+\beta)^3}\;+\;\frac{2}{3(1-\beta)^3} \;=\; \frac{+4\beta(1+\frac{1}{3}\beta^2)}{(1-\beta^2)^3}\,, \tag{186}$$

$$\frac{(1-\beta^2)}{4\xi^4}\Bigg|_{\xi=1-\beta}^{\xi=1+\beta} \;=\; \frac{1-\beta^2}{4}\left(\frac{1}{(1+\beta)^4}-\frac{1}{(1-\beta)^4}\right) \;=\; \frac{-2\beta(1+\beta^2)}{(1-\beta^2)^3}\,, \tag{187}$$

$$\text{altogether} = \frac{(4/3)\beta^3}{(1-\beta^2)^3}, \tag{188}$$

and therefore

$$\oiint d\Omega \frac{\sin^2\theta}{(1-\beta\cos\theta)^5} = \frac{2\pi}{\beta^3}\times\frac{(4/3)\beta^3}{(1-\beta^2)^3} = \frac{8\pi}{3}\times\frac{1}{(1-\beta^2)^3} = \frac{8\pi}{3}\times\gamma^6. \tag{189}$$

Altogether, we find that *the net EM power radiated by a relativistic charged particle accelerating along the same axis as its direction of motion is*

$$P_{\text{rad}}^{\text{net}} = \frac{Q^2\mu_0}{6\pi c}\,\mathbf{a}^2\gamma^6. \tag{190}$$

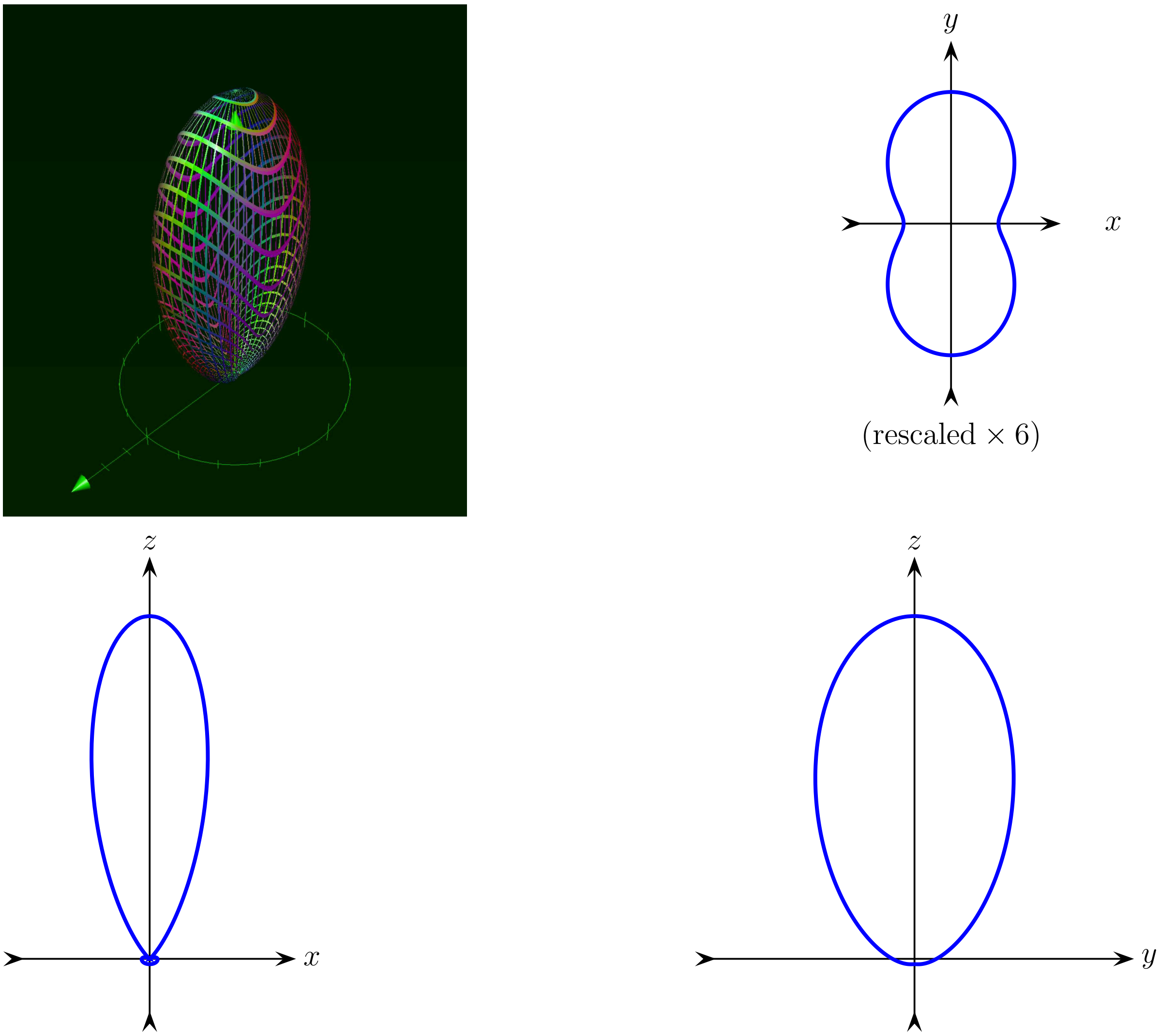
2. A particle moving at constant speed v but in a curved line, so it has acceleration $\perp$ to the velocity vector, For the sake of definiteness, let the acceleration point in the $\hat{\mathbf{x}}$ direction while the velocity points in the $\hat{\mathbf{z}}$ direction. Then, after a bit of algebra and trigonometry (Mathematica helps), we find that

$$|\mathbf{n}\times((\boldsymbol{\beta}-\mathbf{n})\times\mathbf{a})|^2 = \frac{1+\beta^2}{2}(1+\cos^2\theta) - \frac{1-\beta^2}{2}(1-\cos^2\theta)\cos(2\phi) - 2\beta\cos\theta \tag{191}$$

and therefore, the radiation power in any particular direction $\mathbf{n}(\theta,\phi)$ is

$$\frac{dP_{\text{rad}}}{d\Omega} = \frac{Q^2\mu_0}{32\pi^2 c} * \frac{(1+\beta^2)(1+\cos^2\theta) - (1-\beta^2)(1-\cos^2\theta)\cos(2\phi) - 4\beta\cos\theta}{(1-\beta\cos\theta)^5}. \tag{192}$$

This time, the angular distribution of this power is more complicated as it depends on both θ and ϕ. Let me illustrate is with the 3D radiation power diagram for $\beta = 0.6$ and its cross-sections through the xy, xz, and yz planes:



Similar to the $\mathbf{a} \parallel \mathbf{v}$ case, the radiated power of the relativistic particle is strongly peaked in the forward direction, and for the ultra-relativistic particles moving almost as fast as light, almost all power is contained in the narrow forward cone $0 < \theta < O(1/\gamma)$. However, the specific angular profile of this forward cone is slightly different from the $\mathbf{a} \parallel \mathbf{v}$ case:

$$\left(\frac{dP}{d\Omega}\right)_{\perp} \propto \frac{(1-(\gamma\theta)^2)^2 \; + \; 4(\gamma\theta)^2\cos^2\phi}{[1+(\gamma\theta)^2]^5} \tag{193}$$

instead of

$$\left(\frac{dP}{d\Omega}\right)_{\parallel} \propto \frac{(\gamma\theta)^2}{[1+(\gamma\theta)^2]^5}\,. \tag{194}$$

As to the net EM power radiated by the particle accelerating $\perp$ to its velocity, eq. (192) implies

$$P_{\rm rad}^{\rm net} = \frac{Q^2\mu_0\mathbf{a}^2}{32\pi^2c}\oiint d\Omega\frac{(1+\beta^2)(1+\cos^2\theta)\;-\;(1-\beta^2)(1-\cos^2\theta)\cos(2\phi)\;-\;4\beta\cos\theta}{(1-\beta\cos\theta)^5}, \tag{195}$$

where the angular integral evaluates using the same methods we used for the integral (182), so let me be brief and skip some intermediate stages of the calculation:

$$\begin{aligned}
&\oiint d\Omega\frac{(1+\beta^2)(1+\cos^2\theta)\;-\;(1-\beta^2)(1-\cos^2\theta)\cos(2\phi)\;-\;4\beta\cos\theta}{(1-\beta\cos\theta)^5}\\
&\quad=\;2\pi\int\limits_0^\pi d\theta\,\sin\theta\times\frac{(1+\beta^2)(1+\cos^2\theta)\;-\;0\;-\;4\beta\cos\theta}{(1-\beta\cos\theta)^5}\\
&\quad=\;\frac{2\pi}{\beta^3}\int\limits_{1-\beta}^{1+\beta}d\xi\,\frac{(1+\beta^2)\xi^2\;-\;2(1-\beta^2)\xi\;+\;(1-\beta^2)^2}{\xi^5}\\
&\quad=\;\frac{2\pi}{\beta^3}\biggl((1+\beta^2)\times\frac{4\beta}{2(1-\beta^2)^2}\;-\;2(1-\beta^2)\times\frac{6\beta+2\beta^3}{3(1-\beta^2)^3}\\
&\qquad\qquad+\;(1-\beta^2)^2\times\frac{8\beta+8\beta^3}{4(1-\beta^2)^4}\biggr)\\
&\quad=\;\frac{2\pi}{\beta^3}\times\frac{8\beta^3}{3(1-\beta^2)^2}\;=\;\frac{16\pi}{3}\times\gamma^4,
\end{aligned} \tag{196}$$

hence

$$P_{\rm rad}^{\rm net} = \frac{Q^2\mu_0\mathbf{a}^2}{6\pi c}\times\gamma^4. \tag{197}$$

Please note: eq. (197) for the acceleration $\perp$ the velocity has a different power of the Lorentz factor γ from eq. (190) for the acceleration $\|$ to the velocity.

For the general directions of the acceleration and the velocity — at any angle between them — the formula for the angular distribution of the EM radiation becomes too messy two spell out in these notes, let alone to derive it. In fact, when I needed it for myself, I had to use Mathematica to avoid making too many mistakes. Likewise, I use Mathematica to integrate over the 4π directions of radiating and get the net EM power radiated by the relativistic particle. However, Alfred–Marie Liénard did all these calculations by hand more than a century ago, and ended up with a surprisingly simple result, nowadays known as the *Liénard–Larmor formula*

$$P_{\rm rad}^{\rm net} \;=\; \frac{Q^2\mu_0\gamma^6}{6\pi c}\Big(\mathbf{a}^2\;-\;(\beta\times\mathbf{a})^2\Big)\;=\;\frac{Q^2\mu_0}{6\pi c}\Big(\gamma^6\mathbf{a}_\|^2\,+\,\gamma^4\mathbf{a}_\perp^2\Big), \tag{198}$$

where the second equality follows from

$$\mathbf{a}^2\;-\;(\beta\times\mathbf{a})^2\;=\;\mathbf{a}_\|^2\;+\;(1-\beta^2)\mathbf{a}_\perp^2\;=\;\mathbf{a}_\|^2\;+\;\frac{\mathbf{a}_\perp^2}{\gamma^2}\,. \tag{199}$$

Clearly, eqs. (190) and (197) for the power radiated by the particle accelerating $\|$ or $\perp$ to its velocity are special cases of the Liénard–Larmor formula (198).

Example: Synchrotron

A synchrotron is a particle accelerator in which particles follow a fixed circular path through a ring of magnets. The particles travel in bunches, and there is oscillating electric field synchronized with the bunches' motion so it keeps accelerating the particles in the forward direction. At the same time, the magnetic field is adjusted to the particles' momenta so they keep going in the circle of the right radius. Once the particles reach the desired energy, the acceleration stops and the synchrotron acts as a *storage ring*. Unfortunately, the sideways acceleration of the particles moving in a circle makes them radiate — this is known as the *synchrotron radiation* — and the radiation power comes at the expense of the particles' kinetic energies. So to keep the particles moving at a fixed kinetic energy, one has to keep accelerating them forwards, which costs a lot of electric power (and hence money) an causes all kinds of engineering problems.

Specifically, the centripetal acceleration of a particle on a circular path of radius R is

$$a\;=\;\frac{v^2}{R}\;=\;\frac{\beta^2c^2}{R}\,, \tag{200}$$

hence by the Liénard–Larmor formula, the EM power radiated by a particle is

$$P\;=\;\frac{Q^2\mu_0}{6\pi c}\times\gamma^4a^2\;=\;\frac{Q^2\mu_0c^3}{6\pi}\times\frac{\beta^4\gamma^4}{R^2}\,. \tag{201}$$

Thus, over one complete turn around the synchrotron, the particle loses energy

$$\Delta U \;=\; P \times \frac{2\pi R}{\beta c} \;=\; \frac{Q^2 \mu_0 c^2}{3} \times \frac{\beta^3 \gamma^4}{R} \;=\; \frac{Q^2}{3\epsilon_0} \times \frac{\beta^3 \gamma^4}{R}\,. \tag{202}$$

At the same time, the kinetic energy of a relativistic energy is

$$U_{\rm kin} \;=\; (\gamma - 1) \times mc^2 \tag{203}$$

(I shall derive this formula in April), so the fraction of its kinetic energy a particle loses over one turn is

$$\frac{\Delta U}{U_{\rm kin}} \;=\; \frac{Q^2}{3\epsilon_0\, mc^2} \times \frac{\beta^3 \gamma^4}{\gamma - 1} \times \frac{1}{R}, \tag{204}$$

which for an ultra-relativistic particle with $\beta \approx 1$ and $\gamma \gg 1$ may be approximated as

$$\frac{\Delta U}{U_{\rm kin}} \;=\; \frac{Q^2}{3\epsilon_0\, mc^2} \times \frac{\gamma^3}{R} \tag{205}$$

This energy loss is negligible for the present-day proton synchrotrons but can be very problematic for the electron synchrotrons.

Indeed, consider the most powerful proton synchrotron ever built (as of today), the Large Hadron Collider. The LHC is actually a double accelerator where two proton beams collide almost head-on, but each half is a synchrotron, so the energy losses are governed by eq. (205). The LHS tunnel is 27 km long, but 8 km are taken by the straight sections, so the radius of curvature or the remaining circular sections is only 3026 m. The protons have rest energy $m_p c^2 = 938$ MeV, and the LHS accelerates them to energy of $U_{\rm kin} = 6.5$ TeV, thus $\gamma \approx 7000$. Plugging all these numbers into eq. (205), we get

$$\frac{\Delta U}{U_{\rm kin}} \;=\; \frac{e^2}{3\epsilon_0 m_p c^2} \times \frac{(7000)^3}{3026\ {\rm m}} \;\approx\; 7.3 \cdot 10^{-10}, \tag{206}$$

which is indeed negligibly small.

On the other hand, the electrons and the positrons lose much larger fractions of their energies to the synchrotron radiation than the protons, because they have much smaller $mc^2 = 0.512$ MeV factors in the denominator of eq. (205), and also because they are often accelerated to much larger factors of γ. For example, the highest-energy electron-positron collider build thus far was LEP, which used to occupy the same tunnel at CERN that is now used by the LHS. (Thus, same $R = 3026$ m.) At first, LEP 1 accelerated the electrons and the positrons to $U = 45.5$ GeV (which corresponds to $\gamma = 89,000$) to explore the Z^0 particle, but later (LEP 2) the energy was more than doubled to $U = 105$ GeV ($\gamma = 205,000$). Plugging these numbers into eq. (205), we get

$$\frac{\Delta U}{U_{\rm kin}} \;=\; \frac{e^2}{3\epsilon_0 m_e c^2} \times \frac{(89,000)^3}{3026\ {\rm m}} \;\approx\; 2.75 \cdot 10^{-3} \tag{207}$$

for the LEP 1, and

$$\frac{\Delta U}{U_{\rm kin}} \;=\; \frac{e^2}{3\epsilon_0 m_e c^2} \times \frac{(205,000)^3}{3026\ {\rm m}} \;\approx\; 3.36 \cdot 10^{-2} \tag{208}$$

for the LEP 2. At this rate of energy losses, the entire electron's (or positron's) energy had to be replenished every 30 turns around the tunnel. And any synchrotron with a higher rate of energy loss would lose its main advantages over a linear accelerator: the ability to accelerate particles over may turns around the synchrotron rather than over a single pass through a linac, and in a collider to to save the un-collided particles in the storage ring for future collisions.

Note that for a given upper limit on the energy loss fraction, the radius of an electron synchrotron or a circular electron-positron collider scales with the maximal electron energy as

$$R \;\propto\; \gamma^3 \;\propto\; U^3. \tag{209}$$

That's why, the next circular electron-positron colliders currently being planned at CERN and in China are going to have much larger circumference then LEP or LHC, somewhere in the 90 km to 100 km range.

Example: Linear Accelerator

An alternative to a synchrotron accelerator of electrons (or positrons) is a linear accelerator (linac) in which the particle travels in a straight line from the injector to the target. In this case, the particle's acceleration is forward rather than sideways, which at first blush seems to make the Liénard–Larmor radiation worse: unlike $a_\perp^2$ which comes with the factor of γ^4, the $a_\parallel^2$ comes with the larger factor of γ^6. However, the relativistic version of the Newton's 2nd Law comes to our rescue: As we shall learn in April, for a force F parallel to the particle's direction of motion,

$$F \;=\; ma\times\gamma^3, \tag{210}$$

where m is the particle's rest mass. Consequently, in a linear accelerator

$$ma\gamma^3 \;=\; F \;=\; \frac{\text{energy gain}}{\text{length}} \;=\; \frac{mc^2(\gamma_{\text{fin}}-1)}{L}\,, \tag{211}$$

hence

$$a\gamma^3 \;=\; \frac{c^2(\gamma_{\text{fin}}-1)}{L} \;\approx\; \frac{c^2\gamma_{\text{fin}}}{L}\,. \tag{212}$$

Plugging this formula into the Liénard–Larmor equation (198) for the $\parallel$ case, we have

$$P \;=\; \frac{Q^2\mu_0}{6\pi c}\left(a\gamma^3\right)^2 \;=\; \frac{Q^2\mu_0}{6\pi c}\,\frac{c^4\gamma_{\text{fin}}^2}{L^2}\,, \tag{213}$$

which is much less that for a circular accelerator with $R\sim L$. Over the entire time of acceleration, the particle loses energy $\Delta U = P\times t_{\text{flight}}$ where $t_{\text{flight}}\approx L/c$ since most of the time the particle moves at speed close to the speed of light, thus

$$\Delta U \;\approx\; \frac{Q^2\mu_0}{6\pi c}\,\frac{c^4\gamma_{\text{fin}}^2}{L^2}\times\frac{L}{c} \;=\; \frac{Q^2\mu_0 c^2}{6\pi}\times\frac{\gamma_{\text{fin}}^2}{L}\,. \tag{214}$$

As a fraction of the final kinetic energy $U_{\text{fin}} = (\gamma_{\text{fin}}-1)mc^2 \approx \gamma_{\text{fin}}mc^2$ reached by the particle, the energy loss to the EM radiation is

$$\frac{\Delta U}{U_{\text{kin}}} \;=\; \frac{Q^2\mu_0 c^2}{6\pi mc^2}\times\frac{\gamma_{\text{fin}}}{L} \;=\; \frac{Q^2}{6\pi\epsilon_0 mc^2}\times\frac{\gamma_{\text{fin}}}{L}\,. \tag{215}$$

Note that for a given particle species, this fraction is proportional to γ_{fin}/L rather than to γ_{fin}^3/R for the circular accelerators, which makes a tremendous difference for the electron or positron accelerators with $\gamma_{\rm fin} \sim 10^5$: For any existing or currently being planned electron/positron linacs, the energy losses to the Liénard–Larmor radiation are too small to matter.