RECEIVING ANTENNAS

Antennas are used for both transmitting and receiving the EM waves — especially the radio waves and the microwaves, — and often the same pair of antennas is used for transmitting radio signals back and forth in both directions. Thus far in this class we have focused on the transmitting antennas, so in these notes I shall say a few words about the receiving antennas and the antenna reciprocity theorem.

Let me start with the theorem. But first, I need a short-hand notation. Suppose antenna#1 transmits a signal of some frequency ω while antenna#2 receives this signal. In the process, a small fraction of the power transmitted by the antenna#1 is received by the antenna#2; let's denote this fraction

$$\langle 2|1\rangle \stackrel{\rm def}{=} \frac{\text{power received by antenna\#2}}{\text{net power transmitted by antenna\#1}}\,. \tag{1}$$

Likewise, when the antenna#2 transmits the signal while the antenna#1 receives it, let's denote

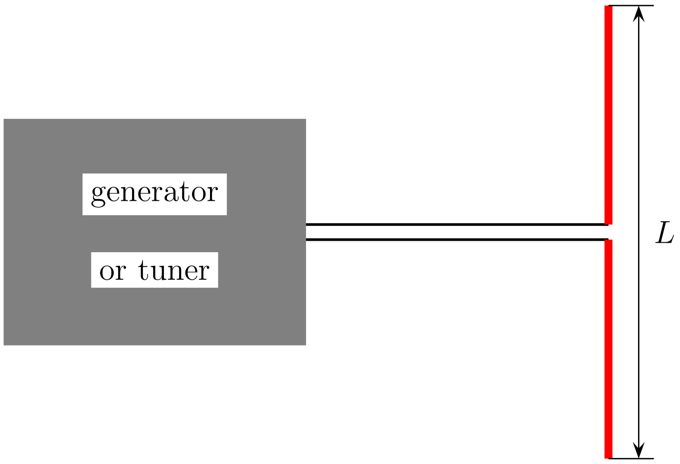
$$\langle 1|2\rangle \stackrel{\rm def}{=} \frac{\text{power received by antenna\#1}}{\text{net power transmitted by antenna\#2}}\,. \tag{2}$$

The **antenna reciprocity theorem** says that *for the same configuration of the two antennas and the same signal frequency,*

$$\langle 1|2\rangle = \langle 2|1\rangle\,. \tag{3}$$

Corollary: *any antenna has exactly the same directivity $D(\mathbf{n})$ when it's used for receiving as when it's used for transmitting.*

In these notes, I am not going to prove the reciprocity theorem or the corollary for general antennas. Instead, I shall simply illustrate them for the simplest kind of antenna, the Hertzian short dipole



$$L \ll \lambda. \tag{4}$$

But first, let me define the antenna's directivity $D(\mathbf{n})$. For a transmitting antenna, the directivity is the ratio of the power (per solid angle) going into the direction $\mathbf{n}$ to its average over all 4π directions,

$$D(\mathbf{n}) \stackrel{\text{def}}{=} \frac{dP}{d\Omega_{\mathbf{n}}} \Bigg/ \frac{P_{\text{net}}}{4\pi}\,. \tag{5}$$

A related term is the antenna's *gain* G, which is the maximal directionality measured in decibels,

$$G = 10\,\lg(D_{\max}). \tag{6}$$

For example, for a short dipole antenna

$$\frac{dP}{d\Omega} = \text{constant} \times \sin^2\theta \quad\Longrightarrow\quad \frac{P_{\text{net}}}{4\pi} = \text{same constant} \times \frac{2}{3}\,, \tag{7}$$

so its directivity is $D(\theta) = \frac{3}{2}\sin^2\theta$. In particular, its maximal directivity is $D_{\max} = \frac{3}{2}$, so its gain is $G = 1.76$ db.

For another example, consider a longer dipole antenna of length $L = \lambda$. As we saw in my notes on long antennas, for this antenna

$$\frac{dP}{d\Omega} = \text{constant} \times \frac{\cos^4(\frac{\pi}{2}\cos\theta)}{\sin^2\theta} \quad\Longrightarrow\quad \frac{P_{\text{net}}}{4\pi} \approx \text{same constant} \times 0.415. \tag{8}$$

The maximal directivity of this antenna (for $\theta = \frac{\pi}{2}$) is therefore $(1/0.415) \approx 2.41$ and the gain is $G = 3.82$ db.

By comparison, phased arrays of multiple antennas can have much higher gains because they manage to concentrate most of the transmitted power into a rather narrow beam. And if the phase shifts of individual antennas are dynamically adjustable, one may turn the radio beam in any direction one likes without any mechanical motions; this is particularly convenient for the radars. For example, the phased-array radar used by the US Aegis cruisers has maximal directivity $D_{\max} = 9300$, *i.e.* gain of almost 40 db.

The microwave dish antennas also have high gains in a narrow range of directions. In general, the maximal directivity of an antenna with a paraboloid dish reflector is

$$D_{\max} \sim \left(\frac{\pi d}{\lambda}\right)^2 \tag{9}$$

where d is the diameter of the dish. For example, a $d = 1$ m dish emitting a $\lambda = 1$ cm signal (frequency = 30 GHz) had $D_{\max} \sim 10^5$, so its gain is 50 db.

Now consider a receiving antenna. The incoming signal comes as an approximately plane wave of intensity — *i.e.*, power per unit area — $I = |\mathbf{S}|$. The antenna captures some of this power, then delivers some fraction of this captured power to the tuner; the rest of the captured power is re-radiated. We are interested in the power that goes to the tuner, so let's define *the effective aperture* of the receiving antenna as

$$A = \frac{\text{power delivered to the tuner}}{\text{intensity of the incoming EM wave}}\,. \tag{10}$$

Note that this ratio has a dimensionality of the area, and the name suggest a window of area A: the power which flows through this window gets delivered to the tuner, while the power that misses the window either keep going in the original direction or else is captured by the antenna and then re-radiated in some other direction.

The effective aperture of a receiving antenna depends on the direction $\mathbf{n}$ and the polarization of the incoming wave as

$$A = A_0 \times D(\mathbf{n}) \times \cos^2\psi, \tag{11}$$

where:

- A_0 is an overall constant. For a properly matched tuner,

$$A_0 \;=\; \frac{\lambda^2}{4\pi}\,. \tag{12}$$

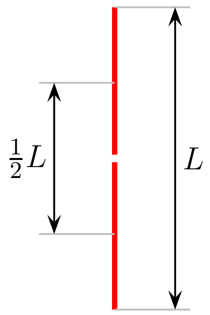
- $D(\mathbf{n})$ is the directivity of the antenna; by the reciprocity theorem, it's exactly the same directivity as for the same antenna transmitting rather than receiving the signal.

- ψ is the angle between the polarization plane of the incoming signal and the antenna's axis.

As an example of eq. (11), consider the short dipole antenna (4). Let the electric field of the incoming EM wave be

$$\mathbf{E}(\mathbf{r},t) \;=\; E_0\mathbf{e}\exp(ik\mathbf{n}\cdot\mathbf{r}\,-\,i\omega t) \tag{13}$$

where $\mathbf{n}(\theta,\phi)$ is the direction of the wave and $\mathbf{e}$ is its unit polarization vector. The two halves of the dipole antenna are at *average displacement* $\frac{1}{2}\mathbf{L}=\frac{1}{2}L\hat{\mathbf{z}}$ from each other,



For $L\ll\lambda$, the electric field (13) is approximately uniform along the antenna's length, so the voltage it induces in the antenna is simply

$$V(t) \;\approx\; \tfrac{1}{2}\mathbf{L}\cdot\mathbf{E}(\mathbf{r}_{\rm center},t) \;=\; \tfrac{1}{2}LE_0(\hat{\mathbf{z}}\cdot\mathbf{e})\exp(ik\mathbf{n}\cdot\mathbf{r}_{\rm center})\exp(-i\omega t). \tag{14}$$

Up to an irrelevant phase, this voltage has amplitude

$$V_0 \;=\; \tfrac{1}{2}LE_0(\hat{\mathbf{z}}\cdot\mathbf{e}) \tag{15}$$

where

$$\hat{\mathbf{z}}\cdot\mathbf{e} \;=\; \sin\theta\times\cos\psi. \tag{16}$$

Indeed, in the coordinate system (x',y',z') where the z' axis is the signal direction $\mathbf{n}$ and the (x', z') plane is the signal's polarization plane, we have

$$\mathbf{e} = (1,0,0)', \qquad \hat{\mathbf{z}} = (\sin\theta\cos\psi, \sin\theta\sin\psi, \cos\theta)', \tag{17}$$

hence $\hat{\mathbf{z}}\cdot\mathbf{e} = \sin\theta \times \cos\psi$ and therefore

$$V_0 = \tfrac{1}{2} L E_0 \sin\theta\cos\psi. \tag{18}$$

This voltage amplitude produces the current through the antenna and the tuner of amplitude

$$I_0 = \frac{V_0}{Z_A + Z_T} \tag{19}$$

where Z_A is the antenna's impedance and Z_T is the input impedance of the tuner. Note: by the reciprocity theorem, the receiving antenna's impedance Z_A is exactly the same as the transmitting radiation impedance, including the radiation resistance

$$\mathrm{Re}(Z_A) = \frac{\pi Z_0}{6} \times \frac{L^2}{\lambda^2} \tag{20}$$

cf. my notes on compact antennas, and the much harder to calculate reactance $\mathrm{Im}(Z_A)$.

Anyway, the current of amplitude (19) flowing through the tuner delivers power

$$P = \tfrac{1}{2}|I_0|^2 \times \mathrm{Re}(Z_T) = \tfrac{1}{2}|V_0|^2 \times \frac{\mathrm{Re}(Z_T)}{|Z_A + Z_T|^2}\,. \tag{21}$$

This power depends on the tuner's input impedance Z_T, and it's maximized for $Z_T = Z_A^*$. Indeed,

$$\frac{\mathrm{Re}(Z_T)}{|Z_A + Z_T|^2} = \frac{\mathrm{Re}(Z_T)}{(\mathrm{Re}(Z_T) + \mathrm{Re}(Z_A))^2 + (\mathrm{Im}(Z_T) + \mathrm{Im}(Z_A))^2}\,, \tag{22}$$

which for any given $\mathrm{Re}(Z_T)$ is maximized for $\mathrm{Im}(Z_T) = -\mathrm{Im}(Z_A)$, hence

$$\frac{\mathrm{Re}(Z_T)}{|Z_A + Z_T|^2} \longrightarrow \frac{\mathrm{Re}(Z_T)}{(\mathrm{Re}(Z_T) + \mathrm{Re}(Z_A))^2} \tag{23}$$

which is maximized for $\mathrm{Re}(Z_T) = \mathrm{Re}(Z_A)$. Altogether, the maximal power is delivered to the tuner with a matching input impedance $Z_T = Z_A^*$, and for such matched tuner

$$\frac{\mathrm{Re}(Z_T)}{|Z_A + Z_T|^2} = \frac{\mathrm{Re}(Z_A)}{(2\,\mathrm{Re}(Z_A))^2} = \frac{1}{4\,\mathrm{Re}(Z_A)}\,, \tag{24}$$

hence

$$P_{\mathrm{tuner}}^{\mathrm{max}} = \frac{|V_0|^2}{8\,\mathrm{Re}(Z_A)}\,. \tag{25}$$

Note: for a matched tuner, only half of the power captured by the antenna goes to the tuner, the other half goes back to the antenna and is re-radiated. Consequently, any antenna connected to a working tuner also re-radiates some of the incoming signal, and this re-radiation can be detected! For example, in the days when TV signal was delivered to the TV set from an antenna rather than from a cable, somebody outside your house could find if your TV was on without watching into your windows. In the countries that had a TV tax (like the UK), that was a useful tool for catching the tax cheaters.

Now let's assume a matched tuner, go back to eq. (25) for the power to the tuner, and plug in eq. (18) for the voltage amplitude and eq. (20) for the antenna radiative resistance:

$$\begin{aligned} P_{\mathrm{tuner}} &= \frac{|V_0|^2}{8\,\mathrm{Re}(Z_A)} \\ &= |E_0|^2 \times (\tfrac{1}{2}L\sin\theta\cos\psi)^2 \Big/ 8 \times \frac{\pi Z_0}{6}\frac{L^2}{\lambda^2} \\ &= \frac{3}{8\pi} \times \frac{|E_0|^2}{2Z_0} \times \lambda^2 \sin^2\theta\cos^2\psi. \end{aligned} \tag{26}$$

Note that the antenna's length L cancels out from this formula. (Although L remains indirectly relevant because it affects the antenna's radiative impedance to which the tuner's input impedance should be matched.) Moreover, the $|E_0|^2/2Z_0$ factor in eq. (26) is the intensity of the incoming EM wave,

$$I = \frac{|E_0|^2}{2Z_0}\,, \tag{27}$$

so the antenna's effective aperture is

$$A = \frac{P_{\mathrm{tuner}}}{I} = \frac{3}{8\pi} \times \lambda^2 \sin^2\theta\cos^2\psi = \frac{\lambda^2}{4\pi} \times \tfrac{3}{2}\sin^2\theta \times \cos^2\psi. \tag{28}$$

In this formula, the $\frac{3}{2}\sin^2\theta$ factor is the directivity of the short dipole antenna, so its effective aperture is

$$A = \left(A_0 = \frac{\lambda^2}{4\pi} \right) \times D(\theta) \times \cos^2 \psi, \tag{29}$$

exactly as promised in eq. (11).

Although I have derived eq. (29) just for the short dipole antennas, it is valid for any other kind of an antenna with a properly matched tuner. But the devil is in the details: Different antennas have different directivity factors $D(\theta, \phi)$, and they also have different radiative impedances Z_A to which the tuner's input impedance should be matched. Fortunately, such details are beyond the scope of our class.