REFLECTION AND REFRACTION OF PLANE EM WAVES

When an electromagnetic wave hits a boundary between different materials, some of the wave's energy is reflected back while the rest continues on through the second material, although the direction of the continuing wave may be somewhat different from the original wave's; this bending of the wave's direction is called the *refraction*. In these notes we shall study the reflection and the refraction of EM waves in some detail, specifically:

- The kinematic relations between the directions of the incident, the refracted, and the reflected waves.
- The dynamical relations between the intensities, the phases, and the polarizations of all the waves.

Directions of the waves

The kinematic relations are independent on the waves' nature: The directions of the incident, the refracted, and the reflected waves obey the same rules for the electromagnetic waves, the sound waves, the wave functions of quantum particles, or any other kinds of a linear harmonic waves. So to keep our analysis as general as possible, consider plane waves of the form

$$\psi(\mathbf{r},t) \;=\; \psi_0 \exp(i\mathbf{k}\cdot\mathbf{r}-i\omega t) \tag{1}$$

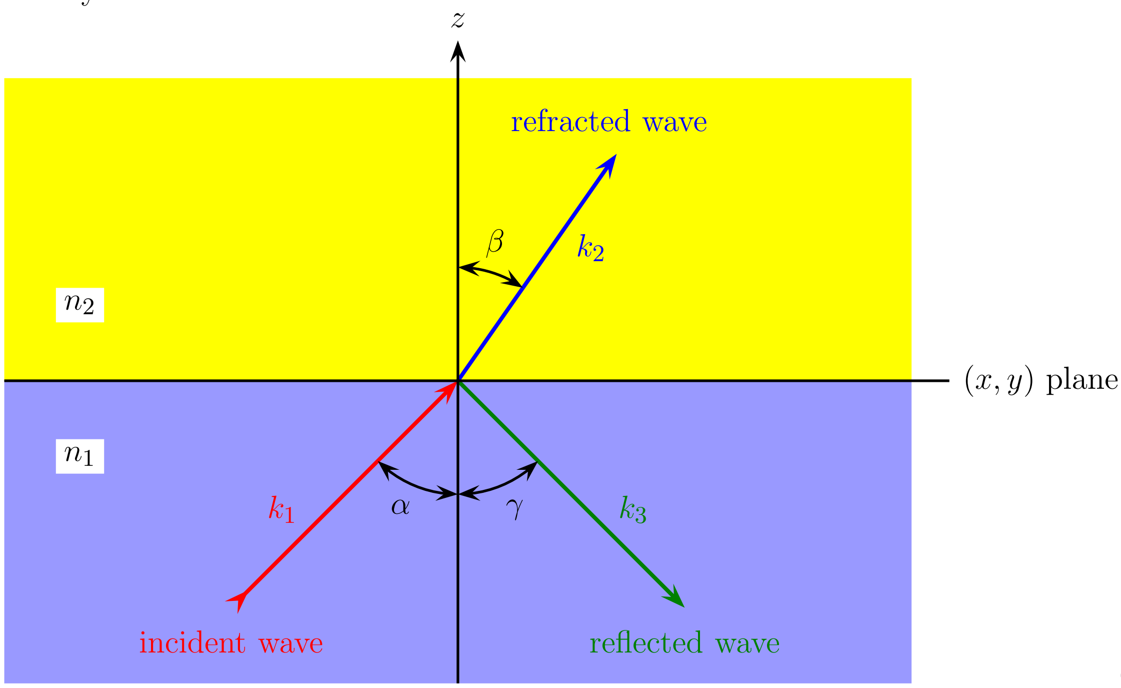
where ψ can be any physical quantity governed by the wave equation. For simplicity, let's assume a flat boundary between two uniform media with different wave speeds $v_1 = c/n_1$ and $v_2 = c/n_2$ but no wave attenuation on either side, thus real n_1 and n_2. The incident wave, the refracted wave, and the reflected wave are all plane waves with the same frequency omega but with different directions of the respective wave vectors $\mathbf{k}$:

$$\psi_{\text{incident}} \;=\; \psi_1 \exp(i\mathbf{k}_1\cdot\mathbf{r}-i\omega t), \tag{2}$$

$$\psi_{\text{refracted}} \;=\; \psi_2 \exp(i\mathbf{k}_2\cdot\mathbf{r}-i\omega t), \tag{3}$$

$$\psi_{\text{reflected}} \;=\; \psi_3 \exp(i\mathbf{k}_3\cdot\mathbf{r}-i\omega t), \tag{4}$$

or graphically



(5)

At the boundary between the two media, the waves satisfy some kind of a linear condition,

$$@z=0,\quad \sum_{j=1,2,3} A_j\times\psi_j\exp(ik_{j,x}x+ik_{j,y}y-i\omega t)\ =\ 0. \tag{6}$$

The coefficients A_1, A_2, and A_3 here may depend on all kind of things — on the nature of the waves in question, on the properties of the two media, on the directions of the waves, on their polarizations (if appropriate), *etc.*, — but they do not depend on the (x, y) coordinates of the boundary point. Therefore, to allow the boundary condition (6) to hold for all (x, y), the (x, y)-dependent phases $\exp(ik_x x + ik_y y)$ must be the same for all three waves — the incident, the refracted, and the reflected, — which means

$$k_{1,x}\ =\ k_{2,x}\ =\ k_{3,x}\quad\text{and}\quad k_{1,y}\ =\ k_{2,y}\ =\ k_{3,y}\,, \tag{7}$$

or in vector notations

$$\mathbf{k}_1^{\|}\ =\ \mathbf{k}_2^{\|}\ =\ \mathbf{k}_3^{\|}\,. \tag{8}$$

The immediate consequence of this relation is that the wave vectors $\mathbf{k}_1$, $\mathbf{k}_2$, $\mathbf{k}_3$ of all 3 waves must lie in the same plane $\perp$ to the boundary; this plane is called *the plane of incidence.*

Now consider the directions of the wave vectors $\mathbf{k}_1$, $\mathbf{k}_2$, $\mathbf{k}_3$ within the plane of incidence. In terms of the angles α, β, and γ on the diagram (5),

$$k_1^{\|} \;=\; k_1 \times \sin\alpha, \quad k_2^{\|} \;=\; k_2 \times \sin\beta, \quad k_3^{\|} \;=\; k_3 \times \sin\gamma,$$

where the magnitudes k_1, k_2, and k_3 follow from the frequency ω and the wave speeds in the respective media, thus

$$k_1 = \frac{\omega n_1}{c}\,, \quad k_2 = \frac{\omega n_2}{c}\,, \quad k_3 = \frac{\omega n_1}{c}\,; \tag{9}$$

note that the reflected wave propagates on the same side as the incident wave but the refracted wave is on the other side. Consequently, eq. (8) tells us that

$$n_1 \times \sin\alpha \;=\; n_2 \times \sin\beta \;=\; n_1 \times \sin\gamma, \tag{10}$$

and therefore:

1. The angle of reflection is equal to the angle of incidence, $\gamma = \alpha$.
2. The angle of refraction β is related to the angle of incidence α by the *Snell's Law,*

$$n_1 \times \sin\alpha \;=\; n_2 \times \sin\beta. \tag{11}$$

Historical note: n is called the *refraction index* of a material precisely because of this formula. Back in early 17$^{\text{th}}$ century when the Snell's law was discovered (or rather re-discovered in Europe, see the Wikipedia page on the subject for the history), people didn't know that light was a wave and had no idea of its speed. The relation $n \times v = c =$ const between the light speed in a material and its refraction index was derived only in 1678, by Christiaan Huygens who developed the wave theory of light.

Total internal reflection.

When a wave coming from the side with the larger refraction index strikes the boundary at a shallow angle, we may have $n_1 \times \sin\alpha > n_2$. In this case, the Snell's Law (11) cannot be satisfied for any angle of refraction β, so the refraction does not happen at all; instead, we have *total internal reflection* of the wave. But this does not mean the complete absence of EM fields behind the boundary; instead, there is the *evanescent wave*

$$\psi_{\text{evanescent}} = \psi_2 \exp(ik_{2x}x + ik_{2y}y) \times \exp(-\kappa_2 z) \times \exp(-i\omega t), \quad \text{for } z > 0 \tag{12}$$

which does not propagate in the z direction but decays with z. The evanescent wave (12) is the analytic continuation of the propagating refracted wave $\exp(i\mathbf{k}\cdot\mathbf{r})$ to the complex wave vector

$$\mathbf{k}_2 = (k_{2x}, k_{2y}, i\kappa_2); \tag{13}$$

this wave obeys the wave equation

$$\left(\nabla^2 + \frac{\omega^2 n_2^2}{c^2}\right)\psi_{\text{evanescent}} = 0 \tag{14}$$

provided

$$(\text{complex } \mathbf{k}_2)^2 = k_{2x}^2 + k_{2y}^2 - \kappa_2^2 = \frac{\omega^2 n_2^2}{c^2}. \tag{15}$$

Since

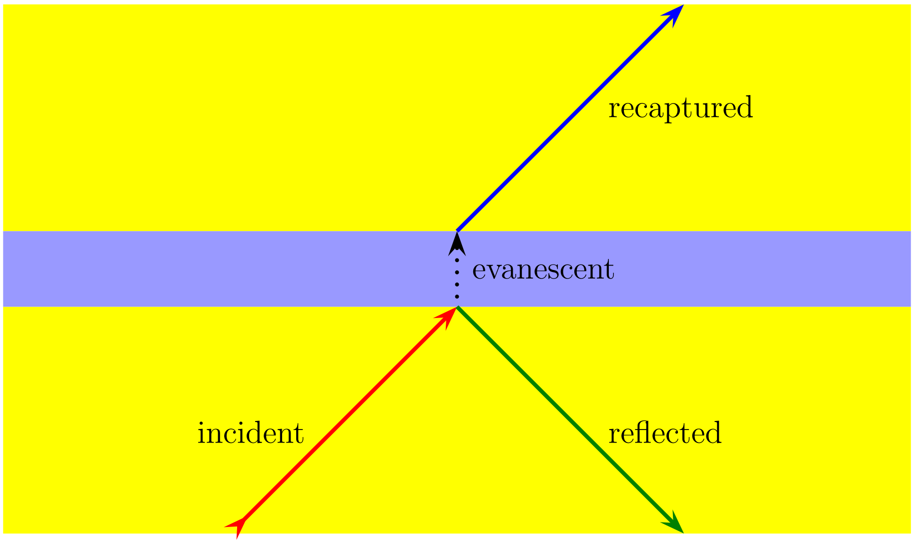
$$k_{2x}^2 + k_{2y}^2 = k_{1x}^2 + k_{1y}^2 = k_1^2 \sin^2\alpha = \frac{\omega^2 n_1^2 \sin^2\alpha}{c^2}, \tag{16}$$

the attenuation rate of the evanescent wave is

$$2\kappa_2 = \frac{2\omega}{c} \times \sqrt{n_1^2 \sin^2\alpha - n_2^2}. \tag{17}$$

For example, the yellow light of frequency $\omega = 3.5 \times 10^{15}\ \text{s}^{-1}$ striking glass-air boundary ($n_{\text{glass}} = 1.5$ vs. $n_{\text{air}} \approx 1$) from the glass side at the $\alpha = 60°$ angle is totally reflected back to the glass, while the evanescent wave in the air attenuate at the rate $2\kappa_2 \approx 19 \times 10^6\ \text{m}^{-1}$: within one micron of depth, it decreases in power by 8 orders of magnitude!

A good way to detect the evanescent wave beyond the total integral reflection is to put another boundary very soon after the first, with a gap between them no wider than a few$/\kappa_2$. For example, take 2 parallel pieces of glass separated by 0.1 μm of air, and let the light coming from one glass piece suffer total integral reflection at the glass-air boundary. This total internal reflection sets up the evanescent wave in the air gap, and once this wave reaches the other glass piece, some of it goes through and becomes an ordinary propagating wave in the other piece:



Reflection and Refraction Coefficients for EM Waves

Now we turn our attention to the dynamical issues of intensities and phases of the reflected and the refracted waves relative to the incident wave. These issues depend on the specific nature of the wave, so let's focus on the plane electromagnetic waves

$$\mathbf{E}_j(\mathbf{r},t) \;=\; \vec{\mathcal{E}}_j \exp(i\mathbf{k}_j\cdot\mathbf{r} - i\omega t), \quad \mathbf{H}_j(\mathbf{r},t) \;=\; \vec{\mathcal{H}}_j \exp(i\mathbf{k}_j\cdot\mathbf{r} - i\omega t), \tag{18}$$

where $j = 1,2,3$ denotes respectively the incident, the refracted, and the reflected waves, while the amplitudes $\vec{\mathcal{E}}_j$ and $\vec{\mathcal{H}}_j$ are complex vectors $\perp$ to the respective wave vectors $\mathbf{k}_j$.

Also, for each wave

$$\vec{\mathcal{H}}_j \;=\; \frac{1}{Z_j}\,\hat{\mathbf{k}}_j\times\vec{\mathcal{E}}_j \tag{19}$$

where $\hat{\mathbf{k}}_j$ is the unit vector in the direction of the $\mathbf{k}_j$ and $Z_j = \sqrt{\mu_j\mu_0/\epsilon_j\epsilon_0}$ is the wave impedance at the appropriate side of the boundary.

The Head-On Case

Let's start with a particularly simple case of the head-on incident wave, $\alpha = 0$. Consequently, the refraction and the reflection angles also vanish, hence

$$\mathbf{k}_1 \;=\; +\frac{n_1\omega}{c}\hat{\mathbf{z}},\quad \mathbf{k}_2 \;=\; +\frac{n_2\omega}{c}\hat{\mathbf{z}},\quad \mathbf{k}_3 \;=\; -\frac{n_1\omega}{c}\hat{\mathbf{z}}. \tag{20}$$

Also, the electric and the magnetic fields of all three waves are $\perp\hat{\mathbf{z}}$ so they lie in the (x,y) plane, which means that both $\mathbf{E}$ and $\mathbf{H}$ fields must be continuous across the boundary at $z=0$,

$$\mathbf{E}(z=+0) \;=\; \mathbf{E}(z=-0),\qquad \mathbf{H}(z=+0) \;=\; \mathbf{H}(z=-0). \tag{21}$$

On the $z<0$ side we have a superposition of the incident and the reflected waves,

$$\mathbf{E}(z,t) \;=\; \vec{\mathcal{E}}_1\exp\left(i\omega\left(\frac{n_1}{c}z-t\right)\right) \;+\; \vec{\mathcal{E}}_3\exp\left(i\omega\left(-\frac{n_1}{c}z-t\right)\right) \;\xrightarrow[z\to-0]{}\; \left(\vec{\mathcal{E}}_1+\vec{\mathcal{E}}_3\right)\exp(-i\omega t) \tag{22}$$

and likewise

$$\mathbf{H}(z,t) \;=\; \vec{\mathcal{H}}_1\exp\left(i\omega\left(\frac{n_1}{c}z-t\right)\right)+\vec{\mathcal{H}}_3\exp\left(i\omega\left(-\frac{n_1}{c}z-t\right)\right) \;\xrightarrow[z\to-0]{}\; (\vec{\mathcal{H}}_1+\vec{\mathcal{H}}_3)\exp(-i\omega t), \tag{23}$$

while on the $z>0$ side we have just the transmitted (refracted) wave

$$\begin{aligned}\mathbf{E}(z,t) \;&=\; \vec{\mathcal{E}}_2\exp\left(i\omega\left(\frac{n_2}{c}z-t\right)\right) \xrightarrow[z\to+0]{} \vec{\mathcal{E}}_2\exp(-i\omega t),\\ \mathbf{H}(z,t) \;&=\; \vec{\mathcal{H}}_2\exp\left(i\omega\left(\frac{n_2}{c}z-t\right)\right) \xrightarrow[z\to+0]{} \vec{\mathcal{E}}_2\exp(-i\omega t).\end{aligned} \tag{24}$$

Consequently, in terms of the amplitudes, the boundary conditions (21) become

$$\begin{aligned} \vec{\mathcal{E}}_1 \;+\; \vec{\mathcal{E}}_3 \;&=\; \vec{\mathcal{E}}_2\,, \\ \vec{\mathcal{H}}_1 \;+\; \vec{\mathcal{H}}_3 \;&=\; \vec{\mathcal{H}}_2\,. \end{aligned} \tag{25}$$

At the same time, the electric and the magnetic amplitudes of the 3 waves are related as

$$\vec{\mathcal{H}}_1 \;=\; +\frac{1}{Z_1}\hat{\mathbf{z}}\times\vec{\mathcal{E}}_1\,, \quad \vec{\mathcal{H}}_2 \;=\; +\frac{1}{Z_2}\hat{\mathbf{z}}\times\vec{\mathcal{E}}_2\,, \quad \vec{\mathcal{H}}_3 \;=\; -\frac{1}{Z_1}\hat{\mathbf{z}}\times\vec{\mathcal{E}}_3\,. \tag{26}$$

so in terms of the electric amplitudes, the magnetic boundary condition (25)(b) becomes

$$\frac{1}{Z_1}\hat{\mathbf{z}}\times\left(\vec{\mathcal{E}}_1 - \vec{\mathcal{E}}_3\right) \;=\; \frac{1}{Z_2}\hat{\mathbf{z}}\times\vec{\mathcal{E}}_2 \tag{27}$$

and hence

$$\vec{\mathcal{E}}_1 \;-\; \vec{\mathcal{E}}_3 \;=\; \frac{Z_1}{Z_2}\vec{\mathcal{E}}_2. \tag{28}$$

Together with the top equation (25)(a), this gives us two linear equations for the three electric amplitudes. Solving these equations, we get the amplitude ratios called the transmission coefficient t and the reflection coefficient r:

$$\begin{aligned} \vec{\mathcal{E}}_2 \;&=\; t\vec{\mathcal{E}}_1 \quad \text{for} \quad t \;=\; \frac{2Z_2}{Z_1+Z_2}\,, \\ \vec{\mathcal{E}}_3 \;&=\; r\vec{\mathcal{E}}_1 \quad \text{for} \quad r \;=\; \frac{Z_1-Z_2}{Z_1+Z_2}\,. \end{aligned} \tag{29}$$

Note that for the head-on EM waves, the transmission and the reflection coefficients do not depend on the wave's polarization: it can be linear in any direction, circular, elliptic, whatever. This follows from the rotational symmetry in the (x,y) plane (which makes for the same t and r for any linear polarization) and the superposition rule (any polarized wave is a sum of two linearly polarized waves in $\perp$ directions).

However, for the waves coming at a non-zero incident angles $\alpha\neq 0$ there is no rotational symmetry so the transmission and the reflection coefficients become polarization-dependent.

However, thanks to the reflection symmetry $\perp$ to the incidence plane, there is a clear polarization eigenbasis for calculating t and r, namely 2 mutually $\perp$ planar polarizations: (1) $\vec{\mathcal{E}}$ normal to the incidence plane, and (2) $\vec{\mathcal{E}}$ within the incidence plane. We shall work these two cases in detail in the next two sections of these notes.

Another noteworthy feature of the transmission and reflection coefficients (29) is that they are real; moreover, t is always positive while r is positive for $Z_2 < Z_1$ but negative for $Z_2 > Z_1$. Consequently, the transmitted wave is always in-phase with the incident wave, while the reflected wave is in-phase for $Z_2 < Z_1$ but of precisely opposite phase for $Z_2 > Z_1$. This behavior is similar to the wave on a string reflected from the place where the string changes its density: if the second string has a lower density — and hence lower wave impedance — then the first string, then the reflected wave is in-phase with the incident phase; but if the second string has a higher density — and hence higher wave impedance — then the reflected wave has an opposite phase from the incident wave.

Next, consider the intensities of the incident, the transmitted, and the reflected waves, *i.e.* the powers (per unit of cross-sectional area) carried by the waves. In general, the intensity of a plane EM wave is

$$I = \frac{|\vec{\mathcal{E}}|^2}{2Z}, \tag{30}$$

so for the waves in question

$$I_1 = \frac{|\vec{\mathcal{E}}_1|^2}{2Z_1}, \quad I_2 = \frac{|\vec{\mathcal{E}}_2|^2}{2Z_2}, \quad I_3 = \frac{|\vec{\mathcal{E}}_3|^2}{2Z_1}, \tag{31}$$

hence

$$\text{the transmissivity } T \stackrel{\text{def}}{=} \frac{I_2}{I_1} = |t|^2 \times \frac{Z_1}{Z_2}, \tag{32}$$

$$\text{and the reflectivity } R \stackrel{\text{def}}{=} \frac{I_3}{I_1} = |r|^2 \times 1. \tag{33}$$

In light of eqs. (29) for the transmission and the reflection coefficients, we get

$$T = \frac{4Z_1 Z_2}{(Z_1+Z_2)^2}, \qquad R = \frac{(Z_1-Z_2)^2}{(Z_1+Z_2)^2}. \tag{34}$$

Note that both the transmissivity and the reflectivity are positive and add up to one,

$$T + R = 1. \tag{35}$$

Physically, this means that the power of the incident wave is divided between the transmitted and the reflected waves, but the net power is conserved. In the absence of absorption, this should always be the case.

I would like to conclude this section with Fresnel's formulae for the transmission and reflection coefficients — and hence the transmissivity and the reflectivity — in terms of the refraction indices n_1 and n_2 of the two media rather than their wave impedances. Fresnel's formulae work only for the non-magnetic media with $\mu \approx 1$, but since most transparent media are non-magnetic, that's OK.

For the non-magnetic media, the refraction index is simply $n \approx \sqrt{\epsilon}$ while the wave impedance

$$Z \approx \frac{Z_0 \approx 377\ \Omega}{\sqrt{\epsilon}} \approx \frac{Z_0}{n}. \tag{36}$$

So is the materials on both sides of the boundary are non-magnetic, we have

$$Z_1 = \frac{Z_0}{n_1} \quad \text{and} \quad Z_2 = \frac{Z_0}{n_2}. \tag{37}$$

Consequently, in terms of the two refraction indices, the transmission and the refraction coefficients become

$$\begin{aligned} t &= \frac{2Z_2}{Z_1 + Z_2} = \frac{2n_1}{n_2 + n_1}, \\ r &= \frac{Z_1 = Z_2}{Z_1 + Z_2} = \frac{n_2 - n_1}{n_2 + n_1}, \end{aligned} \tag{38}$$

and hence the transmissivity and the reflectivity

$$\begin{aligned} T &= \frac{4Z_1 Z_2}{(Z_1 + Z_2)^2} = \frac{4n_2 n_1}{(n_2 + n_1)^2}, \\ R &= \frac{(Z_1 - Z_2)^2}{(Z_1 + Z_2)^2} = \frac{(n_2 - n_1)^2}{(n_2 + n_1)^2}. \end{aligned} \tag{39}$$

For example, at the boundary between the air (with $n_1 \approx 1$) and the glass (with $n_2 \approx 1.5$), the reflectivity is 4% while the transmissivity is 96%.

THE OBLIQUE CASE

Now let's turn our attention to the refraction and reflection of the EM waves hitting the boundary at oblique angles $\alpha \neq 0$. Let's choose our coordinates so the boundary is the (x, y) plane while the incidence plane is (x, z). Then the wave vectors $\mathbf{k}_{1,2,3}$ of all three (incident, refracted, and reflected) waves have $k_y = 0$. In more detail,

$$\begin{aligned}
\mathbf{k}_1 &= n_1(\omega/c) * \big(+\sin_\alpha,\ 0,\ +\cos\alpha\big), \\
\mathbf{k}_2 &= n_2(\omega/c) * \big(+\sin_\beta,\ 0,\ +\cos\beta\big), \\
\mathbf{k}_3 &= n_1(\omega/c) * \big(+\sin_\gamma,\ 0,\ -\cos\gamma\big) \quad \langle\!\langle \text{ for } \gamma = \alpha \,\rangle\!\rangle.
\end{aligned} \tag{40}$$

For simplicity, let us assume the media at both sides of the boundary are non-magnetic and have real refraction indices $n_1 = \sqrt{\epsilon_1(\omega)}$ and $n_2 = \sqrt{\epsilon_2(\omega)}$ at the frequency of the wave, hence

$$Z_1 = \frac{Z_0}{n_1} \quad \text{and} \quad Z_2 = \frac{Z_0}{n_2}\,. \tag{41}$$

Consequently, the relations between the electric and the magnetic amplitudes of the three (incident, refracted, and reflected) waves become

$$Z_0\vec{\mathcal{H}}_1 = n_1\,\hat{\mathbf{k}}_1 \times \vec{\mathcal{E}}_1\,, \quad Z_0\vec{\mathcal{H}}_2 = n_2\,\hat{\mathbf{k}}_2 \times \vec{\mathcal{E}}_2\,, \quad Z_0\vec{\mathcal{H}}_3 = n_1\,\hat{\mathbf{k}}_3 \times \vec{\mathcal{E}}_3\,. \tag{42}$$

We also have relations between amplitudes of different waves stemming from the boundary conditions at $z = 0$. In particular, since the media on both sides of the boundary are non-magnetic, all components of the magnetic field $\mathbf{H}(\hat{\mathbf{z}}, t)$ must be continuous across the boundary. In terms of the incident and the reflected waves at $z \leq 0$ and the refracted wave at $z \geq 0$, this means

$$\mathbf{H}_1(\mathbf{r}, t) + \mathbf{H}_3(\mathbf{r}, t) = \mathbf{H}_2(\mathbf{r}, t) \quad @z = 0. \tag{43}$$

For the three plane waves with all $k_y = 0$ this means

$$\vec{\mathcal{H}}_1 \exp(ik_{1x}x - i\omega t) + \vec{\mathcal{H}}_3 \exp(ik_{3x}x - i\omega t) = \vec{\mathcal{H}}_2 \exp(ik_{2x}x - i\omega t), \tag{44}$$

and since $k_{1x} = k_{2x} = k_{3x}$, all exponentials here are the same and we are left with the amplitude equation

$$\vec{\mathcal{H}}_1 \;+\; \vec{\mathcal{H}}_3 \;=\; \vec{\mathcal{H}}_2\,. \tag{45}$$

As to the electric field, $\epsilon_1 \neq \epsilon_2$ means different boundary conditions for the E_x, E_y components parallel to the boundary versus E_z components normal to the boundary: the E_x, the E_y, and the $E_z \times \epsilon$ must be continuous across the boundary. In terms of the three waves, this means

$$@z=0: \quad \begin{cases} E_{1x}(\mathbf{r},t) \;+\; E_{3x}(\mathbf{r},t) \;=\; E_{2x}(\mathbf{r},t), \\ E_{1y}(\mathbf{r},t) \;+\; E_{3y}(\mathbf{r},t) \;=\; E_{2y}(\mathbf{r},t), \\ n_1^2\Big(E_{1z}(\mathbf{r},t) \;+\; E_{3z}(\mathbf{r},t)\Big) \;=\; n_2^2\,E_{2z}(\mathbf{r},t), \end{cases} \tag{46}$$

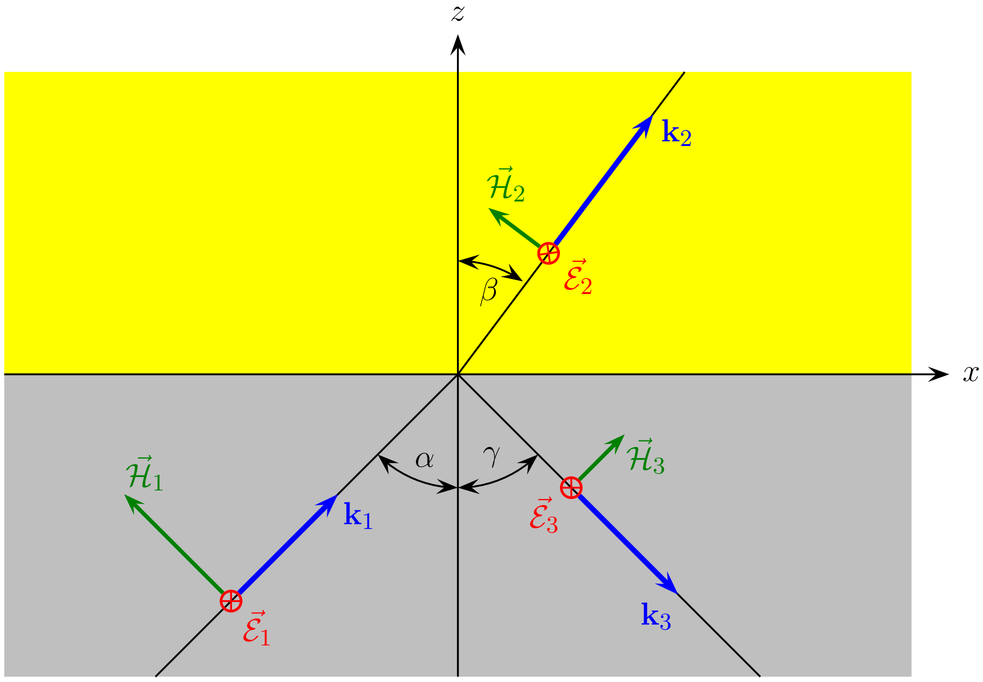
hence in terms of their amplitudes

$$\mathcal{E}_{1,x} \;+\; \mathcal{E}_{3,x} \;=\; \mathcal{E}_{2,x}\,, \quad \mathcal{E}_{1,y} \;+\; \mathcal{E}_{3,y} \;=\; \mathcal{E}_{2,y}\,, \quad n_1^2\big(\mathcal{E}_{1,z}+\mathcal{E}_{3,z}\big) \;=\; n_2^2\,\mathcal{E}_{2,z}\,. \tag{47}$$

The solutions of the boundary conditions (45) and (47) depend on the polarization of the incident wave. Below, we consider 2 orthogonal linear polarizations: (1) the electric field $\vec{\mathcal{E}}_1$ normal to the plane of incidence, and (2) the electric field $\vec{\mathcal{E}}_1$ lying within the plane of incidence. Any other polarization of the incident wave — be it linear, circular, or elliptic — would be a linear combination of these two, so the answers for the appropriate reflected and refracted waves would follow by linearity.

Polarization Normal to the Plane of Incidence

For this polarization, the electric amplitude vector $\vec{\mathcal{E}}_1$ of the incident wave is normal to the plane of incidence, so the reflected and the refracted waves should also have their electric amplitude vectors $\vec{\mathcal{E}}_2$ and $\vec{\mathcal{E}}_3$ be normal to the plane of incidence. On the other hand, the magnetic amplitude vectors $\vec{\mathcal{H}}_1$, $\vec{\mathcal{H}}_2$, and $\vec{\mathcal{H}}_3$ of the three waves must be $\perp$ to the respective electric amplitudes, so they should lie within the plane of incidence. Graphically,



(48)

or in components

$$\vec{\mathcal{E}}_1 = \mathcal{E}_1\big(0,+1,0\big), \tag{49}$$

$$Z_0\vec{\mathcal{H}}_1 = \mathcal{E}_1 n_1\big(-\cos\alpha, 0, +\sin\alpha\big), \tag{50}$$

$$\vec{\mathcal{E}}_2 = \mathcal{E}_2\big(0,+1,0\big), \tag{51}$$

$$Z_0\vec{\mathcal{H}}_2 = \mathcal{E}_2 n_2\big(-\cos\beta, 0, +\sin\beta\big), \tag{52}$$

$$\vec{\mathcal{E}}_3 = \mathcal{E}_3\big(0,+1,0\big), \tag{53}$$

$$Z_0\vec{\mathcal{H}}_3 = \mathcal{E}_3 n_1\big(+\cos\gamma, 0, +\sin\gamma\big). \tag{54}$$

Plugging all these components into the boundary conditions (45) and (47), we arrive at three non-trivial equations

$$\begin{aligned} &[E_y \text{ match}] & \mathcal{E}_1 \;+\; \mathcal{E}_3 &= \mathcal{E}_2\,, \\ &[H_x \text{ match}] & n_1(-\cos\alpha\mathcal{E}_1 \;+\; \cos\gamma\mathcal{E}_3) &= n_2(-\cos\beta\mathcal{E}_2), \\ &[H_z \text{ match}] & n_1(\sin\alpha\mathcal{E}_1 \;+\; \sin\gamma\mathcal{E}_3) &= n_2(\sin\beta\mathcal{E}_2). \end{aligned} \tag{55}$$

However, in light of the reflection law $\gamma = \alpha$ and the Snell's law $n_1 \sin\alpha = n_2 \sin\beta$, the third equation here is equivalent to the first, so there are only 2 independent equations for the two unknown amplitudes $\mathcal{E}_2$ and $\mathcal{E}_3$, namely

$$\mathcal{E}_2 \;-\; \mathcal{E}_3 \;=\; \mathcal{E}_1 \quad\text{and}\quad \frac{n_2\cos\beta}{n_1\cos\alpha}\,\mathcal{E}_2 \;+\; \mathcal{E}_3 \;=\; \mathcal{E}_1\,. \tag{56}$$

The coefficient in the second equation here can be simplified as

$$\frac{n_2\cos\beta}{n_1\cos\alpha} \;=\; \frac{n_2\sin\beta}{n_1\sin\alpha}\bigg/\frac{\tan\beta}{\tan\alpha} \;=\; \frac{\tan\alpha}{\tan\beta} \tag{57}$$

or restated in terms of the angle of incidence α and the refraction indices n_1, n_2 as

$$\frac{n_2\cos\beta}{n_1\cos\alpha} \;=\; \frac{n_2\sqrt{1-\sin^2\beta}}{n_1\cos\alpha} \;=\; \frac{n_2\sqrt{1\;-\;(n_1/n_2)^2\sin^2\alpha}}{n_1\cos\alpha} \;=\; \frac{\sqrt{(n_2/n_1)^2\;-\;\sin^2\alpha}}{\cos\alpha}\,. \tag{58}$$

Hence, solving eqs. (56) for the amplitudes $\mathcal{E}_2$ and $\mathcal{E}_3$ as fractions of the incident wave's amplitude $\mathcal{E}_1$, we get the *transmission coefficient*

$$t \;\stackrel{\text{def}}{=}\; \frac{\mathcal{E}_2}{\mathcal{E}_1} \;=\; \frac{2\tan\beta}{\tan\beta+\tan\alpha} \;=\; \frac{2\cos\alpha}{\cos\alpha\;+\;\sqrt{(n_2/n_1)^2\;-\;\sin^2\alpha}} \tag{59}$$

and the *reflection coefficient*

$$r \;\stackrel{\text{def}}{=}\; \frac{\mathcal{E}_3}{\mathcal{E}_1} \;=\; \frac{\tan\beta-\tan\alpha}{\tan\beta+\tan\alpha} \;=\; \frac{\cos\alpha\;-\;\sqrt{(n_2/n_1)^2\;-\;\sin^2\alpha}}{\cos\alpha\;+\;\sqrt{(n_2/n_1)^2\;-\;\sin^2\alpha}}\,. \tag{60}$$

Note that the transmission coefficient t is always positive — the refracted wave has the same phase as the incident wave, — but the sign of the reflection coefficient seem to depend on the n_2/n_1 ratio as well as the incidence angle α. Actually, it depends only on the n_2/n_1 ratio: For $n_2 > n_1$, r is negative for any α, and the reflected wave has the opposite phase from the incident wave; but for $n_2 < n_1$, r is positive for any α which allows refraction, and the reflected wave has the same phase as the incident wave.

Beside the reflection and the transmission coefficients governing the respective waves' amplitudes, there are related quantities called the *reflectivity* and the *transmissivity* which compare the intensities of the reflected / transmitted waves to that of the incident wave. Or rather, they compare the energy flux densities of the respective waves in the $\pm z$ direction normal to the boundary, thus

$$\text{reflectivity } R \;=\; -\frac{\mathrm{Re}(\vec{\mathcal{E}}_3^* \times \vec{\mathcal{H}}_3)_z}{\mathrm{Re}(\vec{\mathcal{E}}_1^* \times \vec{\mathcal{H}}_1)_z} \tag{61}$$

and

$$\text{transmissivity } T = \;+\frac{\mathrm{Re}(\vec{\mathcal{E}}_2^* \times \vec{\mathcal{H}}_2)_z}{\mathrm{Re}(\vec{\mathcal{E}}_1^* \times \vec{\mathcal{H}}_1)_z}\,; \tag{62}$$

by the energy conservation, they should always add up to one,

$$R \,+\, T \;=\; 1. \tag{63}$$

In terms of the electric amplitudes of the waves,

$$R \;=\; \frac{|\vec{\mathcal{E}}_3|^2}{|\vec{\mathcal{E}}_1|^2} \;=\; |r|^2, \tag{64}$$

while

$$T \;=\; \frac{|\vec{\mathcal{E}}_2|^2 n_2 \cos\beta}{|\vec{\mathcal{E}}_1|^2 n_1 \cos\alpha} \;=\; |t|^2 \times \frac{n_2\cos\beta}{n_1\cos\alpha}\,. \tag{65}$$

where the extra n_2/n_1 factor comes from the magnetic amplitudes while the $\cos\beta/\cos\alpha$ factor comes from projecting the Poynting vector onto the z axis.

For the EM waves polarized normally to the plane of incidence, the reflectivity is

$$R \;=\; \frac{(\tan\alpha - \tan\beta)^2}{(\tan\beta + \tan\alpha)^2} \;=\; \frac{\left(\cos\alpha \,-\, \sqrt{(n_2/n_1)^2 \,-\, \sin^2\alpha}\right)^2}{\left(\cos\alpha \,+\, \sqrt{(n_2/n_1)^2 \,-\, \sin^2\alpha}\right)^2} \tag{66}$$

while the transmissivity is

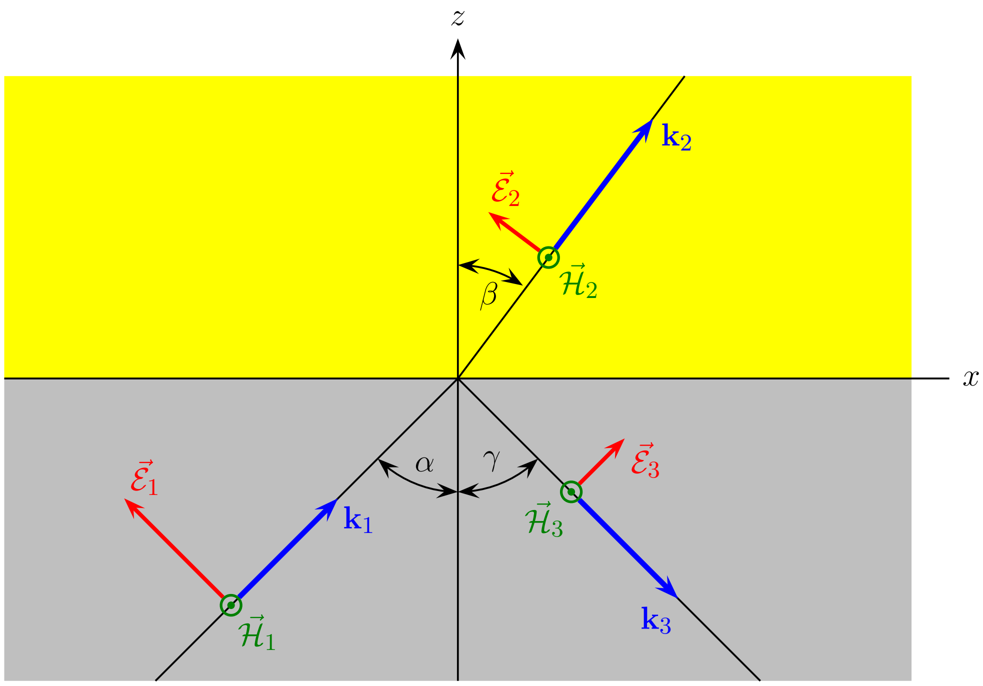
$$T = \frac{4\tan\alpha\tan\beta}{(\tan\beta+\tan\alpha)^2} = \frac{4\cos\alpha\times\sqrt{(n_2/n_1)^2 - \sin^2\alpha}}{\left(\cos\alpha + \sqrt{(n_2/n_1)^2 - \sin^2\alpha}\right)^2}, \tag{67}$$

and its is easy to verify that indeed $R+T=1$. For general incidence angles α, these formulae look somewhat messy, but for the waves hitting the boundary head-on ($\alpha=0$) they reduce to what we saw in the 'head-on' section:

$$R(\alpha=0) = \frac{(n_1-n_2)^2}{(n_1+n_2)^2}, \quad T(\alpha=0) = \frac{4n_1n_2}{(n_1+n_2)^2}. \tag{68}$$

Polarization Within the Plane of Incidence

This time, the incident wave has the electric amplitude vector $\vec{\mathcal{E}}_1$ lying within the plane of incidence, while the magnetic amplitude vector $\vec{\mathcal{H}}_1$ — which is $\perp$ to both $\vec{\mathcal{E}}_1$ and $\mathbf{k}_1$ — must be normal to it. Consequently, the reflected and the refracted waves should also have their magnetic amplitudes $\vec{\mathcal{H}}_3$ and $\vec{\mathcal{H}}_2$ normal to the plane of incidence, while their electric amplitudes $\vec{\mathcal{E}}_3$ and $\vec{\mathcal{E}}_2$ lie within the plane. Graphically,



(69)

so in components

$$\begin{aligned}
\vec{\mathcal{E}}_1 &= \mathcal{E}_1\big(-\cos\alpha, 0, +\sin\alpha\big), & (70)\\
Z_0\vec{\mathcal{H}}_1 &= \mathcal{E}_1\big(0, -n_1, 0\big), & (71)\\
\vec{\mathcal{E}}_2 &= \mathcal{E}_2\big(-\cos\beta, 0, +\sin\beta\big), & (72)\\
Z_0\vec{\mathcal{H}}_2 &= \mathcal{E}_2\big(0, -n_2, 0\big), & (73)\\
\vec{\mathcal{E}}_3 &= \mathcal{E}_3\big(+\cos\gamma, 0, +\sin\gamma\big), & (74)\\
Z_0\vec{\mathcal{H}}_3 &= \mathcal{E}_3\big(0, -n_1, 0\big). & (75)
\end{aligned}$$

Plugging all these components into the boundary conditions (45) and (47), we arrive at three non-trivial equations

$$\begin{aligned}
&[H_y \text{ match}] & n_1(\mathcal{E}_1+\mathcal{E}_3) &= n_2\mathcal{E}_2\,,\\
&[E_x \text{ match}] & -\cos\alpha\mathcal{E}_1 + \cos\gamma\mathcal{E}_3 &= -\cos\beta\mathcal{E}_2\,, \qquad (76)\\
&[E_z \text{ match}] & n_1^2(\sin\alpha\mathcal{E}_1+\sin\gamma\mathcal{E}_3) &= n_2^2\sin\beta\mathcal{E}_2\,,
\end{aligned}$$

but similarly to what we have for the other polarization, the third of these equations becomes equivalent to the first once we use the reflection law $\gamma=\alpha$ and the Snell's law $n_1\sin\alpha = n_2\sin\beta$. This leaves us with two independent equations for two unknown amplitudes $\mathcal{E}_2$ and $\mathcal{E}_3$,

$$\begin{aligned}
\frac{n_2}{n_1}\,\mathcal{E}_2 - \mathcal{E}_3 &= \mathcal{E}_1\,,\\
\frac{\cos\beta}{\cos\alpha}\,\mathcal{E}_2 + \mathcal{E}_3 &= \mathcal{E}_1\,,
\end{aligned} \qquad (77)$$

whose solution gives us the transmission coefficient

$$t = \frac{\mathcal{E}_2}{\mathcal{E}_1} = \frac{2n_1\cos\alpha}{n_1\cos\beta+n_2\cos\alpha} = \frac{2n_1n_2\cos\alpha}{n_1\sqrt{n_2^2-n_1^2\sin^2\alpha}\,+\,n_2^2\cos\alpha} \qquad (78)$$

and the reflection coefficient

$$r = -\frac{\mathcal{E}_3}{\mathcal{E}_1} = \frac{n_1\cos\beta-n_2\cos\alpha}{n_1\cos\beta+n_2\cos\alpha} = \frac{n_1\sqrt{n_2^2-n_1^2\sin^2\alpha}\,-\,n_2^2\cos\alpha}{n_1\sqrt{n_2^2-n_1^2\sin^2\alpha}\,+\,n_2^2\cos\alpha}\,. \qquad (79)$$

The overall minus sign here is a matter of convention; its appropriate for small $\alpha=\gamma$ angles, since in this regime the electric amplitude vectors $\mathcal{E}_3$ and $\mathcal{E}_1$ on the diagram (69) point in the near-opposite directions.

Similar to the other polarization, the transmission coefficient t is always positive — the transmitted (refracted) wave is always in-phase with the incident wave. But this time, the sign of the reflection coefficient r does depend on the incidence angle α: It flips as the α goes through the *Brewster angle*

$$\alpha_b \;=\; \arctan\frac{n_2}{n_1} \tag{80}$$

for which the reflection coefficient vanishes! Geometrically, this Brewster angle is defined by the reflected and the refracted rays being $\perp$ to each other, thus $\beta_b + \gamma_b = 90°$. Indeed, for the incidence angle α_b as in eq. (80), we have

$$\cos\alpha_b \;=\; \frac{1}{\sqrt{1+\tan^2\alpha_b}} \;=\; \frac{n_1}{\sqrt{n_1^2+n_2^2}}\,, \tag{81}$$

$$\sin\alpha_b \;=\; \tan\alpha_b\times\cos\alpha_b \;=\; \frac{n_2}{\sqrt{n_1^2+n_2^2}}\,, \tag{82}$$

$$\begin{aligned}\sin\beta_b \;&=\; \frac{n_1}{n_2}\times\sin\alpha_b \;=\; \frac{n_1}{\sqrt{n_1^2+n_2^2}}\\ &=\; \cos\alpha_b\,,\end{aligned} \tag{83}$$

and hence

$$\beta_b \;=\; 90° \;-\; \alpha_b \;=\; 90° \;-\; \gamma_b\,. \tag{84}$$

And in the context of eq. (79),

$$n_2^2 \;-\; n_1^2\sin^2\alpha_b \;=\; n_2^2 \;-\; \frac{n_1^2 n_2^2}{n_1^2+n_2^2} \;=\; \frac{n_2^4}{n_1^2+n_2^2}\,, \tag{85}$$

$$\begin{aligned} n_1\times\sqrt{n_2^2-n_1^2\sin^2\alpha_b} \;&=\; n_1\times\frac{n_2^2}{\sqrt{n_1^2+n_2^2}} \;=\; n_2^2\times\frac{n_1}{\sqrt{n_1^2+n_2^2}}\\ &=\; n_2^2\times\cos\alpha_b\,,\end{aligned} \tag{86}$$

and therefore

$$[\text{the numerator of eq. (79)}] \;=\; 0 \quad \text{for } \alpha = \alpha_b \quad\Longrightarrow\quad r \;=\; 0. \tag{87}$$

When the un-polarized light is reflected from some surface at the Brewster angle, only the waves polarized normally to the incidence plane are reflected, but the waves polarized withing the plane of incidence do not. Consequently, the light reflected at the Brewster angle is 100%polarized!

For the other angles of reflections, both polarizations get reflected but with different reflection coefficients, so the reflected light is partially polarized. Consequently, an appropriate polarizing filter can suppress the reflected wave more than the incident wave. This effect is exploited by the anti-glare sunglasses. It also helps polarized antennas to focus on to the direct radio or microwave signal while suppressing the echos of that signal reflected from the ground or the ionosphere.

Summary

To summarize these notes, let me write down the transmissivity and the reflectivity of the 2 polarizations of the EM waves:

- For the EM waves polarized normally to the plane of incidence,

$$R \;=\; \frac{\left(\cos\alpha \;-\; \sqrt{(n_2/n_1)^2 \;-\; \sin^2\alpha}\right)^2}{\left(\cos\alpha \;+\; \sqrt{(n_2/n_1)^2 \;-\; \sin^2\alpha}\right)^2}\,, \tag{66}$$

$$T \;=\; \frac{4\cos\alpha \times \sqrt{(n_2/n_1)^2 \;-\; \sin^2\alpha}}{\left(\cos\alpha \;+\; \sqrt{(n_2/n_1)^2 \;-\; \sin^2\alpha}\right)^2}\,. \tag{67}$$

- For the EM waves polarized within the plane of incidence,

$$R \;=\; \frac{\left(\sqrt{(n_2/n_1)^2 - \sin^2\alpha} \;-\; (n_2/n_1)^2\cos\alpha\right)^2}{\left(\sqrt{(n_2/n_1)^2 - \sin^2\alpha} \;-\; (n_2/n_1)^2\cos\alpha\right)^2}\,, \tag{88}$$

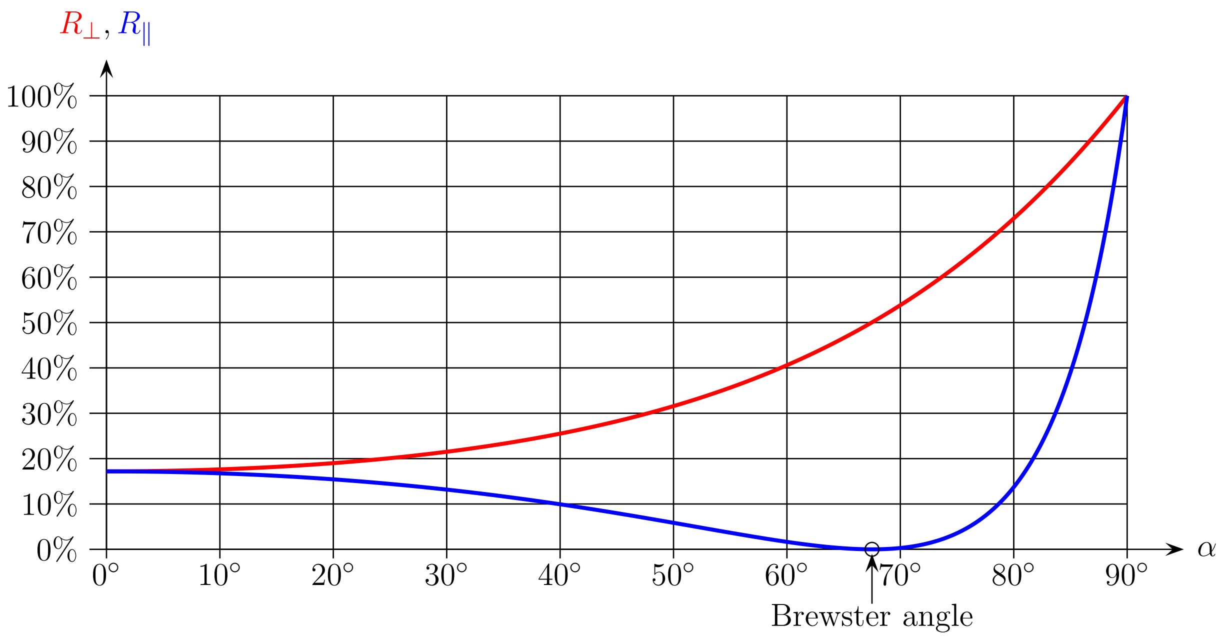
$$T \;=\; \frac{4\sqrt{(n_2/n_1)^2-\sin^2\alpha}\times(n_2/n_1)^2\cos\alpha}{\left(\sqrt{(n_2/n_1)^2-\sin^2\alpha}\;-\;(n_2/n_1)^2\cos\alpha\right)^2}\,. \tag{89}$$

⋆ For the waves hitting the boundary head-on, there is no physical difference between the two polarizations, and indeed both sets of formulae have the same $\alpha=0$ limit:

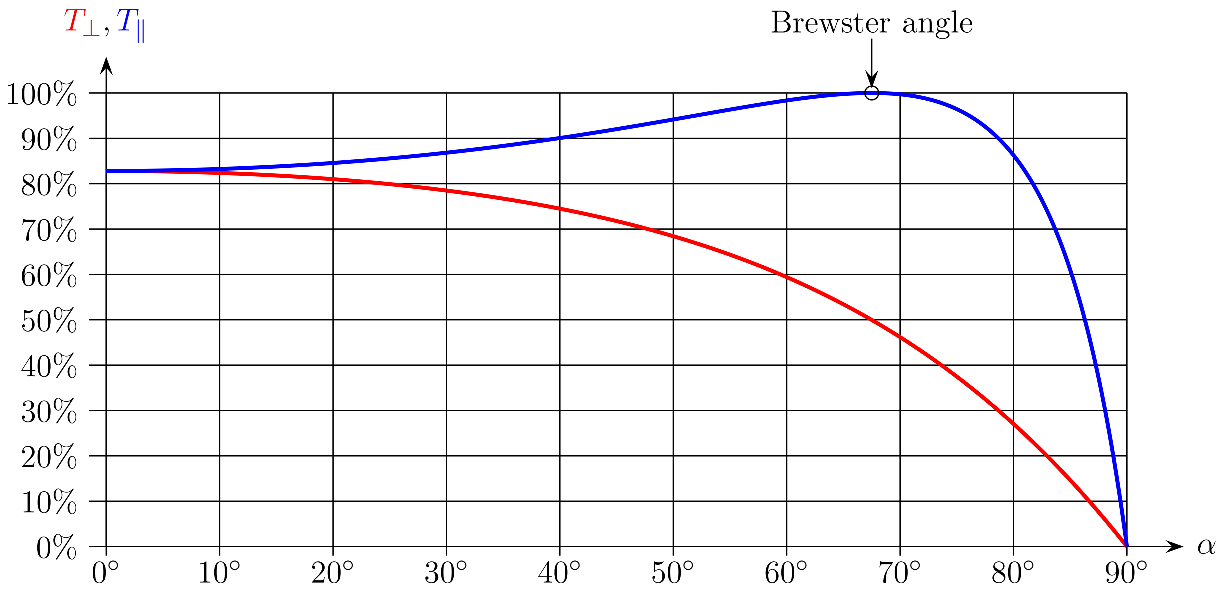
$$R(\alpha=0) \;=\; \frac{(n_1-n_2)^2}{(n_1+n_2)^2}\,,\qquad T(\alpha=0) \;=\; \frac{4n_1n_2}{(n_1+n_2)^2}\,. \tag{90}$$

To illustrate all these formulae, let me plot the reflectivities and the transmissivities of the air-diamond boundary (going from the air with $n_1=1$ to the diamond with $n_2=2.42$) as functions of the incidence angle α.

The reflectivities:



The transmissivities:



Total internal reflection

Finally, let's take another look at the total internal reflection and consider the amplitudes of the reflected and the evanescent wave. Although calling $t = \mathcal{E}_2/\mathcal{E}_1$ the "transmission coefficient" is a misnomer in this case, the Fresnel's equations

$$t_\perp = \frac{2\cos\alpha}{\cos\alpha + \sqrt{(n_2/n_1)^2 - \sin^2\alpha}}, \tag{59}$$

$$t_\parallel = \frac{2(n_2/n_1)\cos\alpha}{(n_2/n_1)^2\cos\alpha + \sqrt{(n_2/n_1)^2 - \sin^2\alpha}}, \tag{78}$$

continue to work, they simply need to be analytically continued to the imaginary square roots. Likewise, the Fresnel's equations for the reflection coefficient

$$r_\perp = \frac{\cos\alpha - \sqrt{(n_2/n_1)^2 - \sin^2\alpha}}{\cos\alpha + \sqrt{(n_2/n_1)^2 - \sin^2\alpha}}, \tag{60}$$

$$r_\parallel = \frac{(n_2/n_1)^2\cos\alpha - \sqrt{(n_2/n_1)^2 - \sin^2\alpha}}{(n_2/n_1)^2\cos\alpha + \sqrt{(n_2/n_1)^2 - \sin^2\alpha}}, \tag{79}$$

work just as well for the total internal reflection after a suitable analytic continuation. But please note: the analytic continuation works for the transmission and reflection coefficients for the amplitudes, but not for the transmissivity $T = |t|^2$ and the reflectivity $R = |r|^2$ for the energy fluxes: taking a mod-square of an amplitude ratio breaks the analytic continuation.

If fact, for the total internal reflection $|r|^2$ is always 1 — the entire energy of the incident wave is reflected back. Indeed, making the analytic continuation of eqs. (60) and (79) explicit, we get

$$r_\perp = \frac{\cos\alpha - i\sqrt{\sin^2\alpha - (n_2/n_1)^2}}{\cos\alpha + i\sqrt{\sin^2\alpha - (n_2/n_1)^2}}, \tag{91}$$

$$r_\parallel = \frac{(n_2/n_1)^2\cos\alpha - i\sqrt{\sin^2\alpha - (n_2/n_1)^2}}{(n_2/n_1)^2\cos\alpha + i\sqrt{\sin^2\alpha - (n_2/n_1)^2}}, \tag{92}$$

which makes it obvious that $|r_\perp| = |r_\parallel| = 1$. On the other hand, for the total internal reflection r is complex rather than real, so the reflected wave has a non-trivial phase shift $\arg(r)$ relative to the incident wave.