ELECTROMAGNETIC WAVES IN CONDUCTORS
Attenuation and Skin Effect

Consider a harmonic EM wave in a conducting material, so besides the displacement current

$$\mathbf{J}_d = \frac{\partial \mathbf{D}}{\partial t} = -i\omega \mathbf{D} = -i\omega\epsilon\epsilon_0 \mathbf{E} \tag{1}$$

there is also the conduction current

$$\mathbf{J}_c = \sigma \mathbf{E}. \tag{2}$$

(The sigma here is the electric conductivity and has nothing to do with surface electric charges.) The Maxwell–Ampere equation combines the conduction and the displacement currents into a single net current

$$\nabla \times \mathbf{H} = \mathbf{J}_{\rm net} = \mathbf{J}_c + \mathbf{J}_d\,, \tag{3}$$

which follows from the electric field as

$$\mathbf{J}_{\rm net} = \sigma_{\rm eff}(\omega)\mathbf{E} \tag{4}$$

where

$$\sigma_{\rm eff}(\omega) = \sigma - i\omega\epsilon_0\epsilon \tag{5}$$

is the complex *effective conductivity*. It is also convenient to define the complex *effective relative permittivity*

$$\epsilon_{\rm eff}(\omega) = \frac{i\sigma_{\rm eff}(\omega)}{\omega\epsilon_0} = \epsilon + \frac{i\sigma}{\omega\epsilon_0}\,, \tag{6}$$

then we may write the Maxwell–Ampere equation as

$$\nabla \times \mathbf{B} = -i\omega \frac{\mu\epsilon_{\rm eff}(\omega)}{c^2}\mathbf{E}\,. \tag{7}$$

Then combining this formula with the other Maxwell equations exactly as we did earlier in class for the non-conducting media (*cf.* my notes on the subject, pages 3–4), we arrive at the EM wave equation

$$\left(\nabla^2 + \frac{n^2(\omega)}{c^2}\omega^2\right)\begin{pmatrix}\mathbf{E}\\ \mathbf{B}\end{pmatrix} = 0 \tag{8}$$

with a complex refraction index

$$n(\omega) = \sqrt{\mu\epsilon_{\rm eff}(\omega)} = \sqrt{\mu\epsilon + \frac{i\mu\sigma}{\epsilon_0\omega}}\,. \tag{9}$$

The general solutions of the wave equation (8) with a complex n are analytic continuations of solutions of the ordinary wave equation. In particular, taking an ordinary plane wave moving in the $+\hat{\mathbf{x}}$ direction and analytically continuing to the complex $n = n_1 + in_2$, we get

$$\mathbf{E}(x,t) = \vec{\mathcal{E}}\exp(ikx - i\omega t), \qquad \mathbf{H}(x,t) = \vec{\mathcal{H}}\exp(ikx - i\omega t), \tag{10}$$

for $k = n\omega/c$, but since n is now complex, we also get complex

$$k = k_1 + ik_2, \qquad k_1 = \frac{n_1\omega}{c}\,, \qquad k_2 = \frac{n_2\omega}{c}\,. \tag{11}$$

Consequently,

$$\exp(ikx) = \exp(ik_1x)\times\exp(-k_2x), \tag{12}$$

which makes the wave (10) an *attenuating wave*

$$\mathbf{E}(x,t) = \vec{\mathcal{E}}\exp(ik_1x - i\omega t)\exp(-k_2x), \qquad \mathbf{H}(x,t) = \vec{\mathcal{H}}\exp(ik_1x - i\omega t)\exp(-k_2x); \tag{13}$$

indeed, as the wave propagates in the $+\hat{\mathbf{x}}$ direction, it also gets weaker and weaker. In particular, the intensity of the wave

$$I = \langle S_x\rangle = \tfrac{1}{2}\,\mathrm{Re}\Big(\vec{\mathcal{E}}^* \times \vec{\mathcal{H}}\Big) = I_0\exp(-2k_2x) \tag{14}$$

decreases with x at the exponential rate

$$\alpha \stackrel{\rm def}{=} -\frac{d\log I}{dx} = 2k_2 = 2n_2\,\frac{\omega}{c}\,. \tag{15}$$

Poor Conductor Limit

As an example of calculating the attenuation rate $\alpha = 2k_2$, consider a poor electric conductor with

$$\sigma \ll \omega\epsilon\epsilon_0\,. \tag{16}$$

For such a material, the conduction current is much weaker than the displacement current; nevertheless, it is this conduction current which leads to attenuation of the EM waves. In terms of the complex effective permittivity, eq. (16) means

$$\operatorname{Im}\epsilon_{\text{eff}} = \frac{\sigma}{\omega\epsilon_0} \ll \epsilon = \operatorname{Re}\epsilon_{\text{eff}}\,. \tag{17}$$

and for such complex numbers

$$\sqrt{\epsilon_{\text{eff}}} \approx \sqrt{\operatorname{Re}\epsilon_{\text{eff}}} + i\frac{\operatorname{Im}\epsilon_{\text{eff}}}{2\sqrt{\operatorname{Re}\epsilon_{\text{eff}}}}\,. \tag{18}$$

Consequently,

$$n_2 = \operatorname{Im}\Big(n = \sqrt{\mu\epsilon_{\text{eff}}}\Big) = \sqrt{\mu}\times\frac{\sigma/\omega\epsilon_0}{2\sqrt{\epsilon}} = \frac{\sqrt{\mu/\epsilon}}{2\epsilon_0}\times\frac{\sigma}{\omega}\,, \tag{19}$$

and hence attenuation

$$\alpha = 2n_2(\omega/c) = \frac{\sqrt{\mu/\epsilon}}{\epsilon_0 c}\times\sigma = Z_{\text{wave}}\times\sigma$$

where

$$Z_{\text{wave}} = \sqrt{\frac{\mu\mu_0}{\epsilon\epsilon_0}} = \frac{\sqrt{\mu/\epsilon}}{c\epsilon_0} \tag{20}$$

is the wave impedance of the material (in the zero-conduction approximation). This, *for a poor conductor, the attenuation rate does not depend on the wave's frequency; instead,*

$$\alpha = Z_{\text{wave}}\times\sigma. \tag{21}$$

For example, consider isopropyl alcohol; it's commonly used for cleaning electronics precisely because of its rather low conductivity $\sigma \approx 6\cdot 10^{-6}$ ℧/m. It also has a rather large dielectric constant $\epsilon \approx 18.6$, so at $\omega = 2\pi \times 1$ MHz we have $\omega\epsilon\epsilon_0 \approx\ 5.3 \cdot 10^{-4}$ ℧/m $\gg \sigma$, so the low-conductivity approximation is valid. Consequently, the attenuation rate of EM waves in a pure isopropyl alcohol is frequency-independent

$$\alpha \ =\ \left(Z_{\text{wave}} = \frac{Z_0}{\sqrt{\epsilon}} \right) \times \sigma \ \approx\ 0.53 \cdot 10^{-3}\ \text{m}^{-1}. \tag{22}$$

In other words, the EM wave can travel through distance

$$L \ =\ \frac{\ln(2)}{\alpha} \ \approx\ 1.3\ \text{km} \ \approx\ 0.82\ \text{mile} \tag{23}$$

before it would lose half of its intensity.

Good Conductor Limit

In the opposite limit of a good conductor with $\sigma \gg \omega\epsilon\epsilon_0$, the conduction current is much larger than the displacement current, and the imaginary part of the effective permittivity ϵ_{eff} is much larger than its real part,

$$\operatorname{Re}\epsilon_{\text{eff}} \ =\ \epsilon \ \ll\ \frac{\sigma}{\omega\epsilon_0} \ =\ \operatorname{Im}\epsilon_{\text{eff}}\,. \tag{24}$$

For such complex numbers

$$\sqrt{\epsilon_{\text{eff}}} \ \approx\ \frac{1+i}{\sqrt{2}} \times \sqrt{|\epsilon_{\text{eff}}|} \ \approx\ \frac{1+i}{\sqrt{2}} \times \sqrt{\operatorname{Im}\epsilon_{\text{eff}}}\,, \tag{25}$$

hence

$$n \ \approx\ \frac{1+i}{\sqrt{2}} \times \sqrt{\mu \times \frac{\sigma}{\omega\epsilon_0}}\,, \tag{26}$$

and therefore

$$n_1 \ =\ n_2 \ =\ \sqrt{\frac{\mu\sigma}{2\omega\epsilon_0}}\,. \tag{27}$$

Or in terms of the complex wave number $k = k_1 + ik_2$,

$$k_1 \ =\ k_2 \ =\ \frac{\omega}{c} \times \sqrt{\frac{\mu\sigma}{2\omega\epsilon_0}} \ =\ \sqrt{\tfrac{1}{2}\mu\mu_0\omega\sigma}\,. \tag{28}$$

Consequently, we have an attenuating wave

$$\mathbf{E}(x,t) \;=\; \vec{\mathcal{E}}\, e^{-i\omega t} e^{(i-1)x/\delta}, \qquad \mathbf{H}(x,t) \;=\; \vec{\mathcal{H}}\, e^{-i\omega t} e^{(i-1)x/\delta} \tag{29}$$

for

$$\delta \;\stackrel{\text{def}}{=}\; \frac{1}{k_2} \;=\; \sqrt{\frac{2\rho}{\mu\mu_0\omega}} \tag{30}$$

(where $\rho = 1/\sigma$ is the materials resistivity). Note that this wave attenuates over a fraction of its wavelength, specifically

$$\frac{1}{\alpha} \;=\; \frac{\delta}{2} \;=\; \frac{1}{2k_2} \;=\; \frac{1}{2k_1} \;=\; \frac{\lambda}{4\pi}\,. \tag{31}$$

The distance δ is called *the skin depth* because of its relation to the *skin effect*: The expulsion of a high-frequency electric current from the interior of a thick conductor. Instead, the current is restricted to the thin subsurface layer of the conductor, *i.e.* the conductor's "skin", hence the name of the effect.

The skin effect follows from the attenuating wave equation (29) for the electric field and the Ohm's Law $\mathbf{J} = \sigma\mathbf{E}$ for the conduction current density. Indeed, the current density $|bj$ follows the electric field: inside the conductor, it attenuates with depth at the exponential rate $1/\delta$, so it does not penetrate to a depth x beyond a few $\times\,\delta$. Instead,

$$\mathbf{J}(x,t) \;=\; \hat{\mathbf{J}}_0 \exp(-i\omega t) * \exp(ix/\delta) \exp(-x/\delta) \tag{32}$$

where $\hat{\mathbf{J}}_0$ is the surface amplitude of the current density and x is the depth inside the conductor, *i.e.* the distance from the surface. According to this formula, for $x \gg \delta$, $\mathbf{J}(x) \approx 0$, so the current indeed flows only through the thin skin $0 < x <$ few $\times\,\delta$ of the conductor.

Note that the skin depth (30) decreases with frequency as $1/\sqrt{\omega}$. For example, consider a copper wire; at room temperature copper has $\rho = 1.68 \cdot 10^{-8}\ \Omega/\mathrm{m}$ and $\mu \approx 1$, hence

$$\delta \;=\; \frac{65.2\ \mathrm{mm}}{\sqrt{f[\text{in Hz}]}}\,. \tag{33}$$

Thus, for the 60 Hz AC current in the power wires, the skin depth is 8.4 mm, but for the 700 MHz frequency used by many cellphones, the skin depth in a copper wire is only 2.5 microns.

Because of the skin effect, the connecting wires have much large impedances to the high-frequency AC currents than their DC resistances. To see how this works, consider a thick conductor, much thicker than the skin depth δ. On the scale of this conductor thickness, the current (32) may be approximated by the surface current of linear density

$$\begin{aligned} \mathbf{K}(t) &= \int_0^\infty dx\, \mathbf{J}(x,t) \\ &= \hat{\mathbf{J}}_0 e^{-i\omega t} \int_0^\infty dx\, e^{(i-1)x/\delta} \\ &= \hat{\mathbf{J}}_0 e^{-i\omega t}\, \frac{\delta}{1-i}\,. \end{aligned} \tag{34}$$

Consequently, the net current through a thick wire is

$$I(t) \;=\; \text{perimeter} \times \frac{\delta}{1-i} \times \hat{J}_0 \times e^{-i\omega t} \tag{35}$$

(assuming the surface current $\mathbf{J}_0$ flows in the long direction of the wire). For example, for a round wire of radius a, the net current $I(t) = \hat{I} e^{-i\omega t}$ has amplitude

$$\hat{I} \;=\; \frac{2\pi a\delta}{1-i}\, \hat{J}_0\,. \tag{36}$$

At the same time, the voltage drop along the wire — as measured along the wire's surface — is

$$V \;=\; L\cdot E_{\text{surface}} \;=\; L\cdot \rho J_{\text{surface}} \;=\; L\rho\, \hat{J}_0 e^{-i\omega t} \;=\; \hat{V} e^{-i\omega t} \tag{37}$$

for the amplitude

$$\hat{V} \;=\; L\rho \hat{J}_0. \tag{38}$$

Comparing this voltage amplitude to the current amplitude (36), we obtain the wire's impedance to the high-frequency current as

$$Z_{\rm HF} = \frac{\hat{V}}{\hat{I}} = \frac{L\rho}{2\pi a\delta} \times (1-i) = \frac{L\rho}{2\pi a\delta} \times (1+j). \tag{39}$$

On the other hand, the same wire's resistance to the DC current is simply

$$R_{\rm DC} = \frac{L\rho}{\pi a^2}\,, \tag{40}$$

hence

$$\frac{Z_{\rm HF}}{R_{\rm DC}} = \frac{(1+j)a}{2\delta}\,, \tag{41}$$

which is a rather large ratio for $a \gg \delta$. For example, a copper wire of diameter $2a = 0.5$ mm has DC resistance of only 0.086 Ω/m, while its AC impedance at 700 MHz becomes much larger $4.3(1+j)$ Ω/m.

Diffusion Equation

Besides have a much larger magnitude than the DC resistance, the HF impedance is also complex rather than real, with a +45° phase (in the EE convention for the sign). In other words, it has not only resistive but also inductive components. And this inductive component militates against sudden changes of the current in the conductor. Indeed, for the currents and EM fields of general (rather than harmonic) time dependence, the currents and the fields obey the *diffusion equation*

$$\begin{aligned} \frac{\partial}{\partial t}\mathbf{J}(\mathbf{r},t) &= \mathcal{D}\nabla^2\mathbf{J}(\mathbf{r},t), \\ \frac{\partial}{\partial t}\mathbf{E}(\mathbf{r},t) &= \mathcal{D}\nabla^2\mathbf{E}(\mathbf{r},t), \\ \frac{\partial}{\partial t}\mathbf{B}(\mathbf{r},t) &= \mathcal{D}\nabla^2\mathbf{B}(\mathbf{r},t), \end{aligned} \tag{42}$$

for the *diffusion coefficient*

$$\mathcal{D} = \frac{1}{\mu\mu_0\sigma}\,. \tag{43}$$

At the same time, the free charges in a conductor flow to the conductor's surface while the bulk charge density decays exponentially as

$$\rho(\mathbf{r},t) \;=\; \rho_0(\mathbf{r})\exp(-t/\tau) \quad \text{for} \quad \tau \;=\; \frac{\epsilon\epsilon_0}{\sigma}\,. \tag{44}$$

Note: the better the conductor, the faster the bulk charges flow to the surface. For example, in the copper metal $\tau \approx 1.5\cdot 10^{-19}$ s, too fast to measure, while in a fused quartz $\tau \approx 3\cdot 10^{+7}$, about a year.

All these formulae stems from the Maxwell equations and the Ohm's Law $\mathbf{J} = \sigma\mathbf{E}$. Indeed, combining Ohm's Law with the Gauss Law and the current continuity equation, we arrive

$$\frac{\partial\rho}{\partial t} \;=\; -\nabla\cdot\mathbf{J} \;=\; -\sigma\nabla\cdot\mathbf{E} \;=\; -\frac{\sigma}{\epsilon\epsilon_0}\nabla\cdot\mathbf{D} \;=\; -\frac{\sigma}{\epsilon\epsilon_0}\,\rho. \tag{45}$$

This is a first-order differential equation WRT time and independent of location $\mathbf{r}$, so its solution is

$$\rho(\mathbf{r},t) \;=\; \rho_0(\mathbf{r})\exp\left(-\frac{\sigma}{\epsilon\epsilon_0}\,t\right) \tag{46}$$

and hence eq. (44).

To derive the diffusion equation for the conducting current $\mathbf{J}$ and the EM fields, let's assume that the initial bulk charge density ρ has already decayed to $\rho = 0$. Consequently, the Maxwell equations become

$$\nabla\cdot\mathbf{E} \;=\; 0, \tag{M1}$$

$$\nabla\times\mathbf{E} \;=\; -\frac{\partial\mathbf{B}}{\partial t}, \tag{M2}$$

$$\nabla\cdot\mathbf{B} \;=\; 0, \tag{M3}$$

$$\nabla\times\mathbf{B} \;=\; \mu\mu_0\sigma\mathbf{E} \;+\; \mu\mu_0\epsilon\epsilon_0\frac{\partial\mathbf{E}}{\partial t} \tag{M4}$$

$$\text{in a good conductor} \quad \approx\; \mu\mu_0\sigma\mathbf{E}. \tag{M4a}$$

Consequently,

$$\begin{aligned}\nabla^2\mathbf{E} \;&=\; \nabla(\nabla\cdot\mathbf{E}) \;-\; \nabla\times(\nabla\times\mathbf{E}) \;=\; \mathbf{0} \;-\; \nabla\times(\nabla\times\mathbf{E})\\ &=\; +\nabla\times\frac{\partial\mathbf{B}}{\partial t} \;=\; \frac{\partial}{\partial t}\big(\nabla\times\mathbf{B}\big)\\ &=\; \mu\mu_0\sigma\frac{\partial\mathbf{E}}{\partial t}\end{aligned} \tag{47}$$

or equivalently

$$\frac{\partial\mathbf{E}}{\partial t} = \frac{1}{\mu\mu_0\sigma}\nabla^2\mathbf{E}, \tag{48}$$

exactly as in eq. (42) for the electric field.

Since the conduction current $\mathbf{J} = \sigma\mathbf{E}$ is proportional to the electric field, it also obeys the similar diffusion equation

$$\frac{\partial\mathbf{J}}{\partial t} = \frac{1}{\mu\mu_0\sigma}\nabla^2\mathbf{J}. \tag{49}$$

Finally, for the magnetic field we have

$$\begin{aligned}\nabla^2\mathbf{B} &= \nabla(\nabla\cdot\mathbf{B}) - \nabla\times(\nabla\times\mathbf{B}) = \mathbf{0} - \nabla\times(\nabla\times\mathbf{B}) \\ &= -\nabla\times\big(\mu\mu_0\sigma\mathbf{E}\big) = -\mu\mu_0\sigma(\nabla\times\mathbf{E}) \\ &= +\mu\mu_0\sigma\frac{\partial\mathbf{B}}{\partial t},\end{aligned} \tag{50}$$

so it also obeys the diffusion equation

$$\frac{\partial\mathbf{B}}{\partial t} = \frac{1}{\mu\mu_0\sigma}\nabla^2\mathbf{B}. \tag{51}$$

As an example of magnetic diffusion, consider a solid metal cylinder surrounded by a solenoidal coil. When we turn on the current in the coil, the surface of the metal cylinder is suddenly exposed to the coil's $\mathbf{H}$ field parallel to the cylinder. But this field cannot instantly penetrate the cylinder; instead, it has to diffuse inward from the surface according to the diffusion equation. This means that at the moment the coil's current I is turned on, we get an equal and opposite counter-current on the cylinder's surface,

$$\mathbf{J}(z,s,\phi) = -K\delta(s-R)\,\mathbf{n}_\phi \quad \text{for } K = \frac{IN}{L}. \tag{52}$$

But as the time passes, this counter-current diffuses inward towards the cylinder's center, and this allows the magnetic field to penetrate the surface and also diffuse inward:

$$\text{for } t>0, \quad \mathbf{J}(\mathbf{r},t) = J(s,t)\,\hat{\boldsymbol{\phi}}, \quad \mathbf{B}(\mathbf{r},t) = B(s,t)\,\hat{\mathbf{z}} \tag{53}$$

for some time-dependent radial profiles $J(s,t)$ and $B(s,t)$.

Alas, solving the diffusion equation for the time dependence of these radial profiles involves Bessel functions and their relatives, and this requires math skill beyond this undergraduate class. Even the simplified 1D problem for the outer layers of the metal cylinder with $(R-s) \ll R$ involves graduate-level math, so let me simply give you the solution: The current profile (for $(R-s) \ll s$) is a Gaussian bell curve, or rather a half $s < R$ of a Gaussian bell curve,

$$J(s,t) \;=\; \frac{2I(N/L)}{\sqrt{\pi}\,a(t)} \times \exp\left(-\frac{(R-s)^2}{a^2(t)}\right) \tag{54}$$

whose width increases with time as

$$a(t) \;=\; \sqrt{4\mathcal{D}\times t}\,, \tag{55}$$

while the magnetic field profile is

$$B(s,t) \;=\; \mu\mu_0(N/L)I \times \big(1 \;-\; \mathrm{erf}((R-s)/a(t))\big) \tag{56}$$

where erf is the *error function* of the Gaussian distribution.

Reflection of EM Waves off a Conductor Surface

Let's go back to the harmonic plane EM waves and consider the reflection of such waves off a surface of a good conductor. For simplicity, let's focus on the head-on case where all waves — the incident, the reflected, and the transmitted — travel $\perp$ to the boundary. Let's that $\perp$ direction to be our z axis with $z=0$ at the boundary. Thus, at $z<0$ there is vacuum (or air approximated as vacuum), and the incident + reflected waves

$$\begin{aligned} \mathbf{E}(z,t) \;&=\; \vec{\mathcal{E}}_i \exp(+ikz-i\omega t) \;+\; \vec{\mathcal{E}}_r \exp(-ikz-i\omega t), \\ \mathbf{H}(z,t) \;&=\; \frac{\hat{\mathbf{z}}}{Z_0}\times\vec{\mathcal{E}}_i \exp(+ikz-i\omega t) \;-\; \frac{\hat{\mathbf{z}}}{Z_0}\times\vec{\mathcal{E}}_r \exp(-ikz-i\omega t), \end{aligned} \tag{57}$$

where $k=\omega/c$ and $Z_0=\sqrt{\mu_0/\epsilon_0}$ is the wave impedance of the free space. On the other hand, at $z>0$ there is some conducting medium with a complex refraction index $n=n_1+in_2$ and a complex wave impedance

$$Z = \frac{\mu Z_0}{n}, \tag{58}$$

so the transmitted wave attenuates with z according to

$$\begin{aligned}
\mathbf{E}(z,t) &= \vec{\mathcal{E}}_t \exp(+in_1kz - i\omega t)\exp(-n_2kz), \\
\mathbf{H}(z,t) &= \frac{n_1 + in_2}{\mu Z_0}\,\hat{\mathbf{z}} \times \vec{\mathcal{E}}_t \exp(+in_1kz - i\omega t)\exp(-n_2kz).
\end{aligned} \tag{59}$$

At the boundary there are no oscillating surface charges or currents, so the tangent electric and magnetic fields **E** and **H** obey simple boundary conditions

$$\mathbf{E}(z \to -0) = \mathbf{E}(z \to +0), \qquad \mathbf{H}(z \to -0) = \mathbf{H}(z \to +0). \tag{60}$$

For the waves (57) and (59), these boundary conditions become

$$\begin{aligned}
\vec{\mathcal{E}}_i + \vec{\mathcal{E}}_r &= \vec{\mathcal{E}}_t\,, \\
\vec{\mathcal{E}}_i - \vec{\mathcal{E}}_r &= \frac{n_1 + in_2}{\mu}\,\vec{\mathcal{E}}_t\,.
\end{aligned} \tag{61}$$

Solving these linear equations, we obtain the reflection and the transmission coefficients

$$\begin{aligned}
r &\stackrel{\text{def}}{=} \frac{\vec{\mathcal{E}}_r}{\vec{\mathcal{E}}_i} = \frac{(n/\mu) - 1}{(n/\mu) + 1}\,, \\
t &\stackrel{\text{def}}{=} \frac{\vec{\mathcal{E}}_t}{\vec{\mathcal{E}}_i} = \frac{2}{(n/\mu) + 1}\,.
\end{aligned} \tag{62}$$

Note: in terms of the complex wave impedances Z_0 for the vacuum and (58) for the conducting material,

$$r = \frac{Z_0 - Z}{Z_0 + Z}\,, \qquad t = \frac{2Z}{Z_0 + Z}\,, \tag{63}$$

exactly as we had earlier in class (*cf.* my notes on refraction and reflection, eq. (29) on page 7), except that for a conducting material, the wave impedance is complex rather than real.

For a complex reflection coefficient r, the reflectivity is

$$R \stackrel{\text{def}}{=} \frac{I_r}{I_i} = |r|^2 = \frac{|Z_0 - Z|^2}{Z_0 + Z|^2}\,, \tag{64}$$

or in terms of the complex refraction index $n = n_1 + in_2$ (and μ, in case the reflector is a magnetic metal like iron),

$$R = \frac{|n-\mu|^2}{|n+\mu|^2} = \frac{n_1^2 + n_2^2 - 2\mu n_1 + \mu^2}{n_1^2 + n_2^2 + 2\mu n_1 + \mu^2} = 1 - \frac{4\mu n_1}{n_1^2 + n_2^2 + 2\mu n_1 + \mu^2}\,. \tag{65}$$

Let's assume the reflector is a good conductor with

$$\epsilon_{\text{eff}} = \epsilon + \frac{i\sigma}{\omega\epsilon_0} \approx i\frac{\sigma}{\omega\epsilon_0} \tag{66}$$

then

$$\sqrt{\epsilon_{\text{eff}}} \approx (1+i) \times \sqrt{\frac{\sigma}{2\omega\epsilon_0}} \tag{67}$$

and hence

$$n = \sqrt{\mu\epsilon_{\text{eff}}} \approx (1+i) \times \sqrt{\frac{\mu\sigma}{2\omega\epsilon_0}} = (1+i) \times \frac{c}{\omega} \times \sqrt{\frac{\mu\mu_0\sigma\omega}{2}} = (1+i) \times \frac{c}{\omega\delta} \tag{68}$$

where δ is the skin depth in the reflecting material. Or in terms of the vacuum wavelength $\lambda = 2\pi c/\omega$ of the EM wave in question,

$$n_1 \approx n_2 \approx \frac{\lambda}{2\pi\delta} \gg 1. \tag{69}$$

Plugging this complex refraction index into eq. (65) for the reflectivity, we get

$$1 - R \approx \frac{4\mu(\lambda/2\pi\delta)}{2(\lambda/2\pi\delta)^2 + 2\mu(\lambda/2\pi\delta) + \mu^2}\,. \tag{70}$$

For a non-magnetic reflector with $\mu \approx 1$, or for a reflector being such a good conductor that $(\lambda/2\pi\delta) \gg \mu$, we may approximate the denominator of this formula by its first term, hence

$$1 - R \approx \frac{4\mu(\lambda/2\pi\delta)}{2(\lambda/2\pi\delta)^2} = \frac{2\mu}{(\lambda/2\pi\delta)} = \frac{4\pi\mu\delta}{\lambda}\,. \tag{71}$$

Example 1: Radio waves of frequency $\omega = 2\pi \times 100$ MHz (in the FM broadcast range) reflecting off the surface of the sea. The sea water has conductivity $\sigma \approx 5$ ℧/m and $\mu \approx 1$, so its skin depth at 100 MHz frequency is

$$\delta = \sqrt{\frac{2}{\mu\mu_0\omega\sigma}} \approx 2.25 \text{ cm} \tag{72}$$

while the vacuum wavelength is $\lambda = c/f \approx 3$ m. Thus, we may treat the sea water as a good conductor at 100 MHz, and its reflectivity (for the head-on radio waves) is

$$R = 1 - \frac{4\pi\delta}{\lambda} \approx 100\% - 9.4\% = 90.6\%. \tag{73}$$

Example 2: For another example, let a microwave beam at frequency 10 GHz reflect off a clean surface of soft iron. In weak magnetic fields, the soft iron is an almost linear ferromagnetic material with a rather high relative permeability $\mu \approx 6000$. It is also a so-so electric conductor for a metal; its conductivity $\sigma \approx 6.7 \cdot 10^6$ ℧/m is about 9 times less than copper's. But because of its high μ, it has a shorter skin depth than copper at the same frequency: For 10 GHz frequency,

$$\delta_{\text{iron}} = \sqrt{\frac{2}{\mu_{\text{iron}}\mu_0\omega\sigma_{\text{iron}}}} \approx 0.025\ \mu\text{m}, \tag{74}$$

compared to $\delta_{\text{copper}} \approx 0.65\ \mu$m. At the same time, the vacuum wavelength of the microwaves in question is $\lambda = 3$ cm, so the ratio

$$\frac{\lambda}{2\pi\delta} = \frac{3 \text{ cm}}{2\pi \times 0.025\ \mu\text{m}} \approx 1.9 \cdot 10^5 \tag{75}$$

is not only large but larger that $\mu_{\text{iron}} \approx 6000$. Consequently, we may use the approximation (71) for the reflectivity:

$$1 - R \approx \frac{4\pi\mu\delta}{\lambda} \approx 6.3\%, \tag{76}$$

thus $R \approx 93.7\%$.

Dielectric and Magnetic Losses

The EM wave in conducting materials attenuate with distance because their energy is dissipated by the conduction current $\mathbf{J} = \sigma\mathbf{E}$, hence power loss

$$P_{\text{loss}} = \iiint \frac{\sigma}{2}\left|\hat{\mathbf{E}}\right|^2 d^3\text{Vol}. \tag{77}$$

But even a perfectly non-conducting dielectric may dissipate the electric energy and cause EM waves to attenuate if the dielectric's polarization lags behind the electric field. Indeed, suppose instead of an instant response $\mathbf{P}(t) = \chi\epsilon_0\mathbf{E}(t)$ of the polarization to the electric field we have

$$\mathbf{P}(t) = \chi\epsilon_0\mathbf{E}(t - \delta t), \tag{78}$$

hence for a harmonic electric field $\mathbf{E}(t) = \hat{\mathbf{E}}e^{-i\omega t}$ the polarization has a different phase, or in terms of complex amplitudes

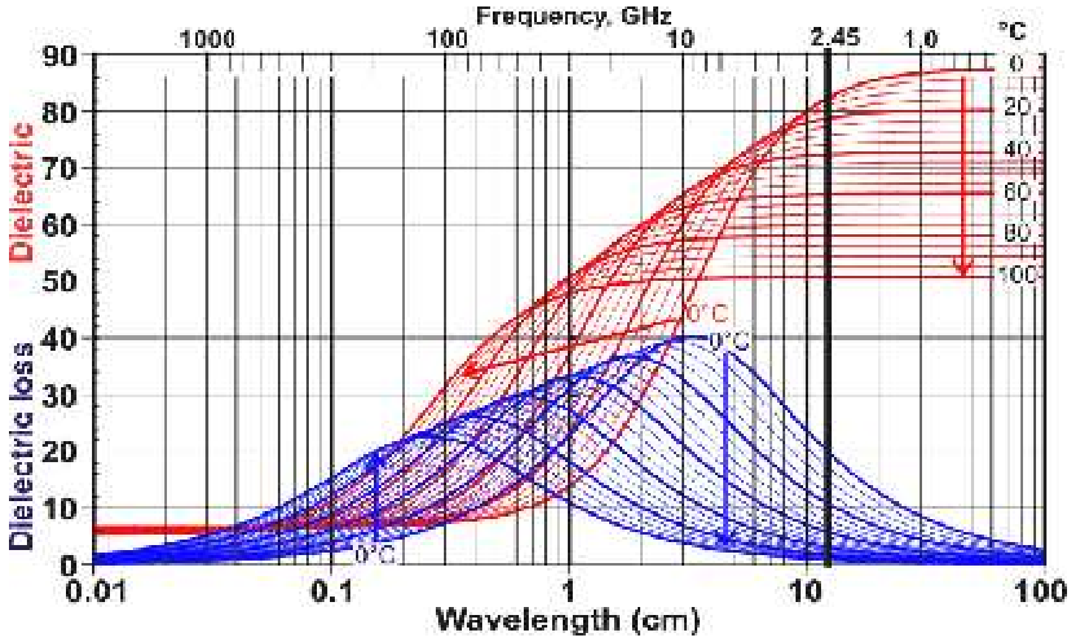
$$\hat{\mathbf{P}} = \chi\epsilon_0 e^{+i\omega\delta t}\hat{\mathbf{E}}. \tag{79}$$

In other words, the dielectric susceptibility becomes complex $\chi \times e^{i\omega\delta t}$, and hence the dielectric constant

$$\epsilon = 1 + \chi \times e^{i\omega\delta t} \tag{80}$$

also becomes complex without any help from the electric conductivity.

In general, the time lag δt depends on the frequency ω, and so does the magnitude of the susceptibility χ, so the frequency dependence of the complex $\epsilon(\omega)$ is more complicated than eq. (80). For example, here is the plot of water's permittivity — both its real part (blue) and imaginary part (red) — as functions of the frequency:



The frequency dependence of the $\text{Re}\,\epsilon(\omega)$ and $\text{Im}\,\epsilon(\omega)$ can be quite complicated, but the imaginary part is always non-negative for any frequency, since the dielectric power loss

$$\frac{\text{power loss}}{\text{volume}} = \tfrac{1}{2}\omega\epsilon_0 \left|\hat{\mathbf{E}}\right|^2 \times \text{Im}(\epsilon) \tag{81}$$

cannot possibly be negative, thus $\text{Im}\,\epsilon \geq 0$. To derive eq.(81), note that the electric work on a dielectric is

$$\delta W = \iiint \mathbf{E}\cdot\delta\mathbf{D}\,d^3\text{Vol}, \tag{82}$$

hence instant electric power per volume

$$\frac{\text{power}}{\text{volume}} = \mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t}\,. \tag{83}$$

Time-averaging this power for the harmonic electric field, we get

$$\frac{\langle\text{power}\rangle}{\text{volume}} = \tfrac{1}{2}\,\text{Re}\Big(\hat{\mathbf{E}}^*\cdot(-i\omega\hat{\mathbf{D}})\Big) = \tfrac{1}{2}\,\text{Re}\Big(\hat{\mathbf{E}}^*\cdot(-i\omega\epsilon\epsilon_0\hat{\mathbf{E}})\Big) = \tfrac{1}{2}\omega\epsilon_0\left|\hat{\mathbf{E}}\right|^2\text{Re}\big(-i\epsilon\big) \tag{84}$$

and hence eq. (81).

When a material has both electric conductivity and $\operatorname{Im}\epsilon > 0$, it has two sources of electric power loss — the Ohmic loss (77) and the dielectric loss (81). Fortunately, both sources combine to

$$\frac{\text{net power loss}}{\text{volume}} = \tfrac{1}{2}\omega\epsilon_0 \left|\hat{\mathbf{E}}\right|^2 \times \operatorname{Im}\bigl(\epsilon_{\text{eff}}\bigr) \tag{85}$$

for

$$\epsilon_{\text{eff}}(\omega) = \epsilon(\omega) + \frac{i\sigma(\omega)}{\omega\epsilon_0}. \tag{86}$$

In this formula, the $\epsilon(\omega)$ itself is complex and frequency dependent, and even the conductivity $\sigma(\omega)$ may be complex and frequency dependent. Indeed, conductivities of metals become complex and frequency-dependent for $\omega \gtrsim 10^{13}\ \text{s}^{-1}$. But regardless of these details, it's the imaginary part of the net $\epsilon_{\text{eff}}(\omega)$ which cases the electric power loss. Likewise, it's the imaginary part of the net $\epsilon_{\text{eff}}(\omega)$ — or rather $\operatorname{Im}\bigl(\sqrt{\epsilon_{\text{eff}}}\bigr)$ — which is responsible for the attenuation rate of the EM waves.

Finally, consider a non-conducting but ferromagnetic material like a ferrite. Similar to a dielectric's polarization lagging behind the electric field which causes it, the magnetization of a ferromagnetic may also lag behind the magnetic field which controls it. Consequently, the magnetic susceptibility and hence the relative permeability μ of the material becomes complex and frequency dependent.

Similar to $\operatorname{Im}\epsilon_{\text{eff}}$ leading to the electric power loss, $\operatorname{Im}\mu$ leads to the magnetic power loss

$$\frac{\text{power loss}}{\text{volume}} = \tfrac{1}{2}\omega\mu_0 \left|\hat{\mathbf{H}}\right|^2 \times \operatorname{Im}\bigl(\mu(\omega)\bigr). \tag{87}$$

And since such loss can never become negative, we always have $\operatorname{Im}\mu \geq 0$ at all frequencies. Also, complex μ leads to attenuation of EM waves via the complex refraction index

$$n(\omega) = \sqrt{\mu(\omega) \times \epsilon_{\text{eff}}(\omega)} \tag{88}$$

Moreover, having $\operatorname{Im}\mu \geq 0$ and $\operatorname{Im}\epsilon_{\text{eff}} \geq 0$ automagically leads to $\operatorname{Im}(n) \geq 0$, so the attenuating plane wave

$$\mathbf{E}, \mathbf{H} \;\propto\; \exp\left(\frac{i\omega}{c}\,\mathrm{Re}(n(\omega))x \;-\; i\omega t\right) \exp\left(-\frac{\omega}{c}\,\mathrm{Im}(n(\omega))x\right) \tag{89}$$

is indeed attenuating rather than gaining strength.