Problem **1**:

For the EM fields (1), the energy density is

$$U = \frac{\epsilon_0}{2}\mathbf{E}^2 + \frac{1}{2\mu_2}\mathbf{B}^2 = \frac{\mu_0^2 c^2 \epsilon_0}{8} K^2(t - \tfrac{|z|}{c}) + \frac{\mu_0}{8} K^2(t - \tfrac{|z|}{c}) = \frac{\mu_0}{4} K^2(t - \tfrac{|z|}{c}), \quad \text{(S.1)}$$

while the Poynting vector is

$$\mathbf{S} = \mathbf{E} \times \mathbf{H} = \frac{1}{\mu_0}\mathbf{E} \times \mathbf{B} = \frac{\mu_0 c}{4} \operatorname{sign}(z)\, K^2(t - \tfrac{|z|}{c})\, \hat{\mathbf{z}}. \quad \text{(S.2)}$$

The Poynting theorem says that

$$\frac{\partial U}{\partial t} + \nabla \cdot \mathbf{S} = \text{should be} = \mathbf{J} \cdot \mathbf{B}, \quad \text{(S.3)}$$

so let's verify it for the fields at hand. The time derivative of the energy density (S.1) is

$$\frac{\partial U}{\partial t} = \frac{\mu_0}{4}\frac{\partial}{\partial t}\big(K^2(t - \tfrac{|z|}{c})\big) = \frac{\mu_0}{2} K(t - \tfrac{|z|}{c})\, K'(t - \tfrac{|z|}{c}), \quad \text{(S.4)}$$

while the divergence of the Poynting vector (S.2) is

$$\begin{aligned} \nabla \cdot \mathbf{S} &= \frac{\mu_0 c}{4}\frac{\partial}{\partial z}\Big(K^2(t - \tfrac{|z|}{c}) \operatorname{sign}(z)\Big) \\ &= \frac{\mu_0 c}{2} K(t - \tfrac{|z|}{c}) \operatorname{sign}(z) * \frac{-\operatorname{sign}(z)}{c} K'(t - \tfrac{|z|}{c}) + \frac{\mu_0 c}{4} K^2(t - \tfrac{|z|}{c}) * 2\delta(z) \\ &= -\frac{\mu_0}{2} K(t - \tfrac{|z|}{c})\, K'(t - \tfrac{|z|}{c}) + \frac{\mu_0 c}{2} K^2(t) * \delta(z). \end{aligned} \quad \text{(S.5)}$$

Altogether, the LHS of the Poynting theorem (S.3) amounts to

$$\frac{\partial U}{\partial t} + \nabla \cdot \mathbf{S} = \frac{\mu_0 c}{2} K^2(t) * \delta(z), \quad \text{(S.6)}$$

while the terms involving $K(t - \tfrac{|z|}{c})\, K'(t - \tfrac{|z|}{c})$ cancel out between $\partial U/\partial t$ and $\nabla \cdot \mathbf{S}$. As to the RHS of the Poynting theorem (S.3),

$$\mathbf{J}(t,z) \;=\; \delta(z)\mathbf{K}(t) \;=\; \delta(z)K(t)\hat{\mathbf{x}} \tag{S.7}$$

while $\mathbf{E}$ at $z=0$ is

$$\mathbf{E}(t,z=0) \;=\; -\frac{\mu_0 c}{2}\,K(t)\hat{\mathbf{x}}, \tag{S.8}$$

hence

$$\mathbf{J}\cdot\mathbf{E} \;=\; -\frac{\mu_0 c}{2}\,K^2(t)\delta(z). \tag{S.9}$$

Comparing this formula to eq. (S.6), we immediately see that indeed,

$$\frac{\partial U}{\partial t} \;+\; \nabla\cdot\mathbf{S} \;=\; \mathbf{J}\cdot\mathbf{E}, \tag{S.10}$$

in perfect agreement with the Poynting theorem.

Problem **2**:

(a) For a normal incidence, the reflection coefficient r is the same for any linear polarization. And since two $\perp$ linear polarizations form a complete basis for all polarizations, the reflected wave has exactly the same polarization as the incident wave, whatever that might be. In our case, the reflected wave is circularly polarized.

(b) For an oblique incidence at a generic incidence angle α_i, there are different reflection coefficients $r_\|$ and $r_\perp$ for the linear polarizations in-plane or normal to the plane of incidence. In terms of this polarization basis, the incident wave has amplitude vector

$$\vec{\mathcal{E}}_i \;=\; \frac{\mathcal{E}_0}{\sqrt{2}}\big(\mathbf{e}_\| \,\pm\, i\mathbf{e}_\perp\big), \tag{S.11}$$

so the reflected wave's amplitude is

$$\vec{\mathcal{E}}_r \;=\; \frac{\mathcal{E}_0}{\sqrt{2}}\big(r_\|\mathbf{e}_\| \,\pm ir_\perp\mathbf{e}_\perp\big). \tag{S.12}$$

For $r_\| \neq r_\perp$ — and also $r_\| \neq 0$ and $r_\perp \neq 0$, as we expect for a generic α_i, — the wave with amplitude (S.12) is elliptically polarized.

Also, for $n_1 < n_2$ there is no total internal reflection, so both reflection coefficients $r_\|$ and $r_\perp$ should be real. Consequently, the two axes of the elliptic polarization (for the reflected wave) correspond to the plane of incidence and the normal to the plane of incidence.

(c) At the Brewster angle $r_\| = 0$ while $r_\perp \neq 0$. Consequently, the reflected wave is linearly polarized $\perp$ to the plane of incidence regardless of the incident wave's polarization. (Unless the incident wave is linearly polarized in plane of incidence, in which case there is no reflection at all.) In particular, for the circularly polarized incident wave, the reflected wave is linearly polarized $\perp$ to the plane of incidence.

(d) For the total internal reflection, both reflection coefficients $r_\|$ and $r_\perp$ have unit magnitudes, $|r_\|| = |r_\perp| = 1$, but they generally have different phases, $\arg(r_\|) \neq \arg(r_\perp)$. Consequently, the reflected wave (S.12) is elliptically rather than circularly polarized.

Problem **3**:

The magnetic field of a time-dependent current $\mathbf{J}(\mathbf{r}, t)$ is given by the Jefimenko equation,

$$\mathbf{B}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int d^3\mathrm{Vol}' \left(\frac{\mathbf{J}(\mathbf{r}', t_{\mathrm{ret}}) \times \mathbf{n}}{\mathcal{R}^2} + \frac{\dot{\mathbf{J}}(\mathbf{r}', t_{\mathrm{ret}}) \times \mathbf{n}}{c\mathcal{R}} \right), \tag{S.13}$$

which for a current $I(t)$ in a thin wire becomes

$$\mathbf{B}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \oint (d\mathbf{r}' \times \mathbf{n}) \left(\frac{I(t_{\mathrm{ret}})}{\mathcal{R}^2} + \frac{\dot{I}(t_{\mathrm{ret}})}{c\mathcal{R}} \right). \tag{S.14}$$

Since the wire in question is a circular ring and we are only interested in the $\mathbf{B}$ field at the ring's center $\mathbf{r} = 0$, we have $\mathcal{R} \equiv a$, $t_{\mathrm{ret}} = t - (a/c)$, hence

$$I(t_{\mathrm{ret}}) = I_0 \exp(-i\omega(t - \tfrac{a}{c})) = I_0 \exp(i\omega a/c) \exp(-i\omega t) \tag{S.15}$$

and therefore

$$\dot{I}(t_{\mathrm{ret}}) = -i\omega I_0 \exp(i\omega a/c) \exp(-i\omega t). \tag{S.16}$$

Finally, along the ring

$$d\mathbf{r}' \times \mathbf{n} \;=\; a\hat{\mathbf{z}}\,d\phi \tag{S.17}$$

where $\hat{\mathbf{z}}$ is the unit vector $\perp$ to the ring's plane.

Plugging all these formulae into the Jefimenko equation (S.14) gives us

$$\begin{aligned}\mathbf{B}(\mathbf{0},t) \;&=\; \frac{\mu_0}{4\pi}\oint d\phi\, a\hat{\mathbf{z}}\left(\frac{I_0 e^{i\omega a/c}e^{-i\omega t}}{a^2} + \frac{-i\omega I_0 e^{i\omega a/c}e^{-i\omega t}}{ca}\right)\\ &=\; \frac{\mu_0\hat{\mathbf{z}}}{2}\, I_0 e^{i\omega a/c}e^{-i\omega t}\left(\frac{1}{a} + \frac{-i\omega}{c}\right).\end{aligned} \tag{S.18}$$

Or in terms of $k = \omega/c$,

$$\mathbf{B}(\mathbf{0},t) \;=\; \frac{\mu_0 I_0}{2a}\big(1 - ika\big)e^{ika}e^{-i\omega t}\hat{\mathbf{z}}. \tag{S.19}$$

<u>Problem **4**</u>:

An oscillating magnetic charge M leads to an oscillating magnetic dipole moment

$$m(t) \;=\; AM\cos(\omega t). \tag{S.20}$$

This oscillating dipole moment radiates EM waves at power

$$P \;=\; \frac{\mu_0}{6\pi c^3}\,(\ddot{m})^2 \;=\; \frac{\mu_0}{6\pi c^3}\Big(-\omega^2 AM\,\cos(\omega t)\Big)^2 \;=\; \frac{\mu_0\omega^4 A^2 M^2}{6\pi c^3}\,\cos^2(\omega t). \tag{S.21}$$

Time-averaging this power, we get

$$\langle P\rangle \;=\; \frac{\mu_0\omega^4 A^2 M^2}{6\pi c^3} * \frac{1}{2} \;=\; \frac{\mu_0\omega^4 A^2 M^2}{12\pi c^3}\,. \tag{S.22}$$

It remains to plug in the monopole's magnetic charge M into this formula. By the Dirac's quantization condition, in MKSA units

$$e \times M \;=\; \frac{2\pi\hbar}{\mu_0} \times \text{an integer}, \tag{S.23}$$

so let's assume the minimal allowed magnetic charge

$$M \;=\; \frac{2\pi\hbar}{\mu_0 e}\,. \tag{S.24}$$

Plugging this charge into eq. (S.22) for the radiated EM power, we get

$$P \;=\; \frac{\mu_0\omega^4 A^2}{12\pi c^3} \times \left(\frac{2\pi\hbar}{\mu_0 e}\right)^2 \;=\; \frac{\pi\hbar^2}{3\mu_0 e^2 c^3} \times \omega^4 A^2. \tag{S.25}$$

<u>Problem **5**</u>:

Motion at constant acceleration was explained in class in detail, *cf.* my notes on spacetime geometry, especially eqs. (77) and (82). In particular, for uniform acceleration starting from rest, the Earth time t is related to the probe's proper time τ as

$$\frac{at}{c} \;=\; \sinh\frac{a\tau}{c} \tag{S.26}$$

while the displacement from the starting point is

$$\Delta x \;=\; \frac{c^2}{a}\left(\cosh\frac{a\tau}{c} \,-\, 1\right). \tag{S.27}$$

For $a/c = 1/\text{year}$, these formulae simplify to

$$t[\text{in years}] \;=\; \sinh\tau[\text{inyears}] \tag{S.28}$$

and

$$\Delta x[\text{inlightyears}] \;=\; \cosh\tau[\text{inyears}] \;-\; 1. \tag{S.29}$$

So the 3 on-board years of acceleration time τ correspond to

$$t = \Big(\sinh(3) \approx 10.018\Big) \text{ years} \tag{S.30}$$

of Earth time, during which the probe flies through the distance of

$$\Delta x = \Big(\cosh(3) - 1 \approx 9.068\Big) \text{ lightyears.} \tag{S.31}$$

The deceleration phase takes another 3 on-board years corresponding to $t \approx 10.02$ Earth years, and during this time the probe flies through another $\Delta x \approx 9.07$ lightyears. So by the time the probe stops in the planetary system X, its distance from the Earth is

$$L = 2 \times \Delta x = 18.136 \text{ lightyears.} \tag{S.32}$$

Also the probe arrives to the system X at time

$$T_{\text{arrival}} = 2 \times t = 20.036 \text{ years} \tag{S.33}$$

by the Earth clock. At that time, the probe start sending data back to Earth, but the signal takes an extra time

$$\Delta T_{\text{signal}} = \frac{L}{c} = 18.136 \text{ years} \tag{S.34}$$

to arrive, so the first data arrive to Earth after

$$T_{\text{arrival}} + \Delta T_{\text{signal}} = 38.172 \text{ years} = 38 \text{ years and 63 days.} \tag{S.35}$$

from launch. So if the probe is launched on January 1, 2050, the data starts arriving on March 4, 2088.

Problem **6**:

In the rest frame of the conducting material

$$\rho_r \;=\; 0, \qquad \mathbf{J}_r \;=\; \sigma\mathbf{E}_r\,, \tag{S.36}$$

regardless of the magnetic field $\mathbf{B}_r$. In terms of the electric and magnetic fields in the lab frame where the conductor moves at velocity $\mathbf{v}$,

$$\mathbf{E}_r^{\|} \;=\; \mathbf{E}^{\|}\,, \qquad \mathbf{E}_r^{\perp} \;=\; \gamma\mathbf{E}^{\perp} \;+\; \gamma\mathbf{v}\times\mathbf{B}, \tag{S.37}$$

hence the rest-frame current

$$\mathbf{J}_r^{\|} \;=\; \sigma\mathbf{E}^{\|}\,, \qquad \mathbf{E}_r^{\perp} \;=\; \gamma\sigma\big(\mathbf{E}^{\perp} \;+\; \mathbf{v}\times\mathbf{B}\big). \tag{S.38}$$

Lorentz-transforming this current — and also $\rho_r = 0$ — to the lab frame, we get

$$\begin{aligned} \mathbf{J}^{\|} \;&=\; \gamma\mathbf{J}_r^{\|} \;-\; \gamma\mathbf{v}\rho_r \;=\; \gamma\sigma\mathbf{E}^{\|} \;-\; 0, \\ \mathbf{J}^{\perp} \;&=\; \mathbf{J}_r^{\perp} \;=\; \gamma\sigma\big(\mathbf{E}^{\perp} \;+\; \mathbf{v}\times\mathbf{B}\big). \end{aligned} \tag{S.39}$$

Together, these two formulae amount to

$$\mathbf{J} \;=\; \gamma\sigma\big(\mathbf{E} \;+\; \mathbf{v}\times\mathbf{B}\big), \tag{2}$$

quod erat demonstrandum.