Problem **1**:

(a) The force between two halves of the charged sphere obtains from the EM stress tensor $\overleftrightarrow{T}$ as the integral

$$\mathbf{F} = -\iint \overleftrightarrow{T} \cdot \mathbf{d^2a} \tag{S.1}$$

over any complete surface separating the two hemispheres. The obvious choice of such a surface is the equatorial plane. For that plane, the area vector $\mathbf{d^2a}$ always points in the z direction, so eq. (S.1) becomes

$$F^i = -\iint T^{iz}\, d^2a. \tag{S.2}$$

Moreover, by symmetry of the problem we know the net force points in the z direction, so all we need is

$$F^z = -\iint T^{zz}\, d^2a. \tag{S.3}$$

Next, in the absence of magnetic field

$$T^{ij} = \epsilon_0 \Big(E^i E^j - \tfrac{1}{2}\mathbf{E}^2 \delta_{ij} \Big), \tag{S.4}$$

in particular

$$T^{zz} = \epsilon_0 \Big((E^z)^2 - \tfrac{1}{2}\mathbf{E}^2 \Big). \tag{S.5}$$

Furthermore, the electric field of the whole charged sphere always points radially from the center. Hence, within the equatorial plane $E^z = 0$, which means

$$T^{zz} = -\tfrac{1}{2}\epsilon_0 \mathbf{E}^2. \tag{S.6}$$

Plugging this stress tensor component into eq. (S.3) for the force, we immediately obtain

$$F^z = + \iint \tfrac{1}{2}\epsilon_0 \mathbf{E}^2\, d^2a. \tag{1}$$

(b) The electric field of a uniformly charged sphere is

$$\mathbf{E} = \frac{Q}{4\pi\epsilon_0}\frac{\mathbf{n}}{r^2} \tag{S.7}$$

outside the sphere (for $r > R$) and $\mathbf{E} = 0$ inside the sphere. Thus, in polar coordinates (r, ϕ) for the equatorial plane,

$$\mathbf{E}^2(r,\phi) = \begin{cases} (Q/4\pi\epsilon_0 r^2)^2 & \text{for } r > R, \\ 0 & \text{for } r < R. \end{cases} \tag{S.8}$$

Consequently, the integral in eq. (1) evaluates to

$$\begin{aligned} F^z &= \iint \frac{\epsilon_0}{2}\mathbf{E}^2\, d^2a = \iint \frac{\epsilon_0}{2}\mathbf{E}^2\, r\, dr\, d\phi \\ &= \frac{2\pi\epsilon_0}{2}\int_0^\infty \mathbf{E}^2(r)\, r\, dr = \frac{2\pi\epsilon_0}{2}\int_R^\infty \left(\frac{Q}{4\pi\epsilon_0 r^2}\right)^2 r\, dr \\ &= \frac{Q^2}{16\pi\epsilon_0}\int_R^\infty \frac{r\, dr}{r^4} = \frac{Q^2}{16\pi\epsilon_0^2} \times \frac{1}{2R^2} \\ &= \frac{Q^2}{32\pi\epsilon_0 R^2}\,. \end{aligned} \tag{S.9}$$

Problem **2**:

(a) The plasma in the ionosphere becomes transparent for $\omega > \omega_p$, and its refraction index depends on the wave's frequency as

$$n_p(\omega) = \sqrt{1 - \frac{\omega_p^2}{\omega^2}}. \tag{S.10}$$

The radio wave hits the ionosphere from below, coming from the ion-less upper atmosphere with $n_a \approx 1$. Note that $n_a > n_p$, so there is total internal reflection when

$$n_a \times \sin\theta \; > n_p \,. \tag{S.11}$$

In light of eq. (S.10) for the plasma and $n_a \approx 1$, this condition amounts to

$$\sin\theta \; > \; \sqrt{1 - \frac{\omega_p^2}{\omega^2}}\,, \tag{S.12}$$

which is mathematically equivalent to

$$\sin^2\theta \; = \; 1 \; - \; \cos^2\theta \; > \; 1 \; - \; \frac{\omega_p^2}{\omega^2} \quad \Longrightarrow \quad \cos\theta \; < \; \frac{\omega_p}{\omega} \tag{S.13}$$

and hence

$$\omega \; < \; \frac{\omega_p}{\cos\theta}\,. \tag{2}$$

(b) The plasma frequency follows from the electron density N_e in the plasma — in our case, the ionosphere, — as

$$\omega_p \; = \; \sqrt{\frac{N_e e^2}{\epsilon_0 m_e}}\,. \tag{S.14}$$

Thus, for $N_e \approx 1.0 \cdot 10^{12}$ electrons per cubic meter, we have

$$\omega_p^2 \; \approx \; (1.0 \cdot 10^{12}\ \mathrm{m}^{-3}) \times (2\pi \times 9)\ \mathrm{Hz})^2 \cdot \mathrm{m}^3 \; = \; \left(2\pi \times 9 \cdot 10^6\ \mathrm{Hz}\right)^2 \tag{S.15}$$

and hence $\omega_p \; = \; 2\pi \, \times \, 9 \; MHz$. Or in terms of the cyclic plasma frequency $f_p \; = \; \omega_p/2\pi$, $f_p \approx 9$ MHz.

Finally, at the incidence angle (between the radio wave and the vertical) $\alpha \approx 72^\circ$, we have $\cos\alpha \approx 0.31$, so the total internal reflection happens for

$$f \; < \; \frac{f_p}{\cos\alpha} \; \approx \; \frac{9\ \mathrm{MHz}}{0.31} \; = \; 29\ \mathrm{MHz} \; \approx \; 30\ \mathrm{MHz}. \tag{S.16}$$

The radio wave with a lower frequency would be totally reflected back to the ground, and that's what allows for the long-range radio broadcasts.

Problem **3**:

(a) The instantaneous Poynting vector is $\mathbf{S} = \mathbf{E} \times \mathbf{H}$, do for a harmonic wave like (3), the time averaged Poynting vector is

$$\langle \mathbf{S} \rangle \;=\; \tfrac{1}{2}\,\mathrm{Re}\big(\mathbf{E}^* \times \mathbf{H}\big) \tag{S.17}$$

For the attenuating wave (3),

$$\mathbf{E}^* \times \mathbf{H} \;=\; E_0^* H_0\, \exp(-2k_2 z)\, \big(\hat{\mathbf{x}} \times \hat{\mathbf{y}} \;=\; \hat{\mathbf{z}}\big), \tag{S.18}$$

hence

$$\langle \mathbf{S} \rangle \;=\; \tfrac{1}{2}\,\mathrm{Re}\big(E_0^* H_0\big)\, \exp(-2k_2 z)\hat{\mathbf{z}}\,. \tag{S.19}$$

Finally, the divergence:

$$\nabla \cdot \big(\exp(-2k_2 z)\hat{\mathbf{z}}\big) \;=\; \frac{\partial}{\partial z} \exp(-2k_2 z) \;=\; -2k_2 \exp(-2k_2 z), \tag{S.20}$$

hence

$$\nabla \cdot \langle \mathbf{S} \rangle \;=\; \tfrac{1}{2}\,\mathrm{Re}\big(E_0^* H_0\big) * (-2k_2)\exp(-2k_2 z). \tag{S.21}$$

(b) The magnetic amplitude of a plane wave is related to its electric amplitude as

$$H_0 \;=\; \frac{n}{\mu Z_0}\, E_0 \tag{S.22}$$

where Z_0 is the wave impedance of the vacuum, $\mu = 1$ for the non-magnetic material, and n is the *complex* refraction index of the conducting material,

$$n \;=\; \sqrt{\epsilon_{\mathrm{eff}} \;=\; \epsilon + \frac{i\sigma}{\omega\epsilon_0}}\,. \tag{S.23}$$

Thus,

$$\mathrm{Re}\big(E_0^* H_0\big) \;=\; \mathrm{Re}\left(E_0^*\, \frac{n}{Z_0}\, E_0\right) \;=\; \frac{|E_0|^2}{Z_0}\, \mathrm{Re}(n), \tag{S.24}$$

and consequently eq. (S.21) for the divergence of the Poynting vector becomes

$$\nabla\cdot\langle\mathbf{S}\rangle \;=\; -\frac{|E_0|^2}{2Z_0}*\mathrm{Re}(n)*2k_2\exp(-2k_2z). \tag{S.25}$$

Next, the attenuation rate $2k_2$ obtains from

$$k_2 \;=\; \mathrm{Im}(k) \;=\; \mathrm{Im}(n)\,\frac{\omega}{c}\,, \tag{S.26}$$

so in eq. (S.25)

$$\mathrm{Re}(n)*2k_2 \;=\; \mathrm{Re}(n)*2\,\mathrm{Im}(n)*\frac{\omega}{c} \;=\; \mathrm{Im}(n^2)*\frac{\omega}{c}\,. \tag{S.27}$$

But

$$\mathrm{Im}(n^2) \;=\; \mathrm{Im}\big(\epsilon_{\mathrm{eff}}\big) \;=\; \frac{\sigma}{\omega\epsilon_0}\,, \tag{S.28}$$

hence

$$\mathrm{Re}(n)*2k_2 \;=\; \frac{\sigma}{\omega\epsilon_0}*\frac{\omega}{c} \;=\; \frac{\sigma}{\epsilon_0 c} \;=\; \sigma Z_0\,. \tag{S.29}$$

Plugging this formula into eq. (S.25), we arrive at

$$\nabla\cdot\langle\mathbf{S}\rangle \;=\; -\tfrac{1}{2}\sigma\,|E_0|^2\exp(-2k_2z). \tag{4}$$

Quod erat demonstrandum.

(c) The instantaneous power loss density is $p=\mathbf{E}\cdot\mathbf{J}$, so for the harmonic EM wave the time-average power loss density is

$$\langle p\rangle \;=\; \tfrac{1}{2}\,\mathrm{Re}\big(\mathbf{E}^*\cdot\mathbf{J}\big). \tag{S.30}$$

The $\mathbf{J}$ here is the conduction current, so by the Ohm's Law $\mathbf{J}=\sigma\mathbf{E}$, hence

$$\langle p\rangle \;=\; \tfrac{1}{2}\,\mathrm{Re}\big(\mathbf{E}^*\cdot\sigma\mathbf{E}\big) \;=\; \tfrac{1}{2}\sigma\,|\mathbf{E}|^2\,, \tag{S.31}$$

as we take the conductivity σ to be real. For the wave (3), eq. (S.31) becomes

$$\langle p\rangle \;=\; \tfrac{1}{2}\sigma\,|E_0|^2\exp(-2k_2z). \tag{S.32}$$

Now, the Poynting theorem — which is the local version of the work-energy theorem for the EM fields — requires

$$\frac{\partial u}{\partial t}\ +\ \nabla\cdot\mathbf{S}\ +\ p\ =\ 0 \tag{S.33}$$

for the instant EM energy density $u_{\rm em}$, EM energy flux $\mathbf{S}$, and EM power density p. Likewise, the time-averaged energy density, energy flux, and power density of an EM wave should obey the similar equation

$$\frac{\partial\langle u\rangle}{\partial t}\ +\ \nabla\cdot\langle\mathbf{S}\rangle\ +\ \langle p\rangle\ =\ 0. \tag{S.34}$$

Now let's check it for the wave in question. By inspection of eqs. (4) and (S.32) we see that

$$\nabla\cdot\langle\mathbf{S}\rangle\ +\ \langle p\rangle\ =\ 0. \tag{S.35}$$

At the same time, the wave attenuates in space but not in time, so the time-averaged energy density $\langle u\rangle$ stays constant, thus

$$\frac{\partial\langle u\rangle}{\partial t}\ =\ 0. \tag{S.36}$$

Altogether, the wave (3) indeed obeys the Poynting equation (S.34). *Quod erat demonstrandum.*

Problem **4**:

(a) The phase velocity and the group velocity of a light wave follow from the refraction index of the material as

$$\frac{c}{v_{\rm phase}}\ =\ n,\qquad \frac{c}{v_{\rm group}}\ =\ n\ +\ \omega\frac{dn}{d\omega}\ =\ n\ -\ \lambda\frac{dn}{d\lambda}\,. \tag{S.37}$$

for the Cauchy formula approximation (5),

$$\begin{aligned} n\ &=\ A\ +\ \frac{B}{\lambda^2}\,,\\ n\ -\ \lambda\frac{dn}{d\lambda}\ &=\ A\ +\ \frac{B}{\lambda^2}\ -\ \lambda\times\frac{-2B}{\lambda^3}\ =\ A\ +\ \frac{3B}{\lambda^2}\,. \end{aligned}$$

For the flint glass in question and the cyan light with $\lambda = 0.500\ \mu$m, these formulae yield

$$n \approx 1.7817, \qquad n - \lambda\frac{dn}{d\lambda} \approx 1.8891, \tag{S.38}$$

hence

$$v_{\rm phase} = 168\,260\ \text{km/s}, \qquad v_{\rm group} = 158\,670\ \text{km/s}. \tag{S.39}$$

(b) As explained in my notes on dispersion, eq. (47) on page 10, the highest rate of pulses that can travel distance L through a dispersive medium without merging up is

$$\nu_{\max} = \sqrt{\frac{c}{L} \Big/ \left|\frac{dn}{d\omega} + 2\omega\frac{d^2n}{d\omega^2}\right|}. \tag{S.40}$$

Rewriting the Cauchy formula (5) as

$$n(\omega) = A + \frac{B}{(2\pi c)^2} \times \omega^2, \tag{S.41}$$

we get

$$\frac{dn}{d\omega} = 2\frac{B}{(2\pi c)^2} \times \omega\,, \tag{S.42}$$

$$\omega\frac{d^2n}{d\omega^2} = 2\frac{B}{(2\pi c)^2} \times \omega\,, \tag{S.43}$$

hence

$$\frac{dn}{d\omega} + 2\omega\frac{d^2n}{d\omega^2} = 6\frac{B}{(2\pi c)^2} \times \omega = \frac{6B}{2\pi c\lambda}\,, \tag{S.44}$$

and therefore

$$\nu_{\max} = \sqrt{\frac{c}{L} \times \frac{2\pi c\lambda}{6B}}\,. \tag{S.45}$$

For the fling glass in question and the cyan light,

$$\frac{2\pi\lambda}{6B} \approx 39\ (\mu\text{m})^{-1} = 39\cdot 10^6\ \text{m}^{-1}\,, \tag{S.46}$$

hence for $L = 1$ m,

$$\nu_{\max} = c \times \sqrt{39\cdot 10^6}\ \text{m}^{-1} = 1.87\cdot 10^{12}\ \text{s}^{-1}. \tag{S.47}$$