
Problem **1**:

(a) The vector potential $\mathbf{A}$ points in the $\hat{\mathbf{z}}$ direction but does not depend on z, hence $\nabla\cdot\mathbf{A} = 0$, so the potentials (1) obey the Coulomb gauge condition. Also, the scalar potential V is time-dependent, thus

$$\frac{1}{c^2}\frac{\partial V}{\partial t} + \nabla\cdot\mathbf{A} = \frac{Dz}{c^2} + 0 \neq 0, \tag{S.1}$$

so the potentials (1) do NOT obey the Landau gauge condition.

(b) The electric field is

$$\mathbf{E} = -\nabla V - \frac{\partial\mathbf{A}}{\partial t} = -Dt\hat{\mathbf{z}} - \mathbf{0} = -Dt\hat{\mathbf{z}} \tag{S.2}$$

while the magnetic field is

$$\begin{aligned} \mathbf{B} = \nabla\times\mathbf{A} &= \frac{D}{4c^2}\left(\hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y}\right)\times(x^2+y^2)\hat{\mathbf{z}} \\ &= \frac{D}{4c^2}\Big(2x(\hat{\mathbf{x}}\times\hat{\mathbf{z}} = -\hat{\mathbf{y}}) + 2y(\hat{\mathbf{y}}\times\hat{\mathbf{z}} = +\hat{\mathbf{x}})\Big) = -\frac{D}{2c^2}\,s\hat{\boldsymbol{\phi}}. \end{aligned} \tag{S.3}$$

(c) The two homogeneous Maxwell equations

$$\begin{aligned} \nabla\cdot\mathbf{B} &= 0, \\ \nabla\times\mathbf{E} &= -\frac{\partial\mathbf{B}}{\partial t}, \end{aligned} \tag{S.4}$$

automagically follow from the very existence of the vector and scalar potentials. The Gauss Law

$$\nabla\cdot\mathbf{E} = \frac{\rho}{\epsilon_0} = 0 \quad \text{when } \rho = 0 \tag{S.5}$$

is trivially true for the electric field (S.2). Finally, the Maxwell–Ampere equation

$$\nabla\times\mathbf{B} = \mu_0\mathbf{J} + \frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t} = \frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t} \quad \text{when } \mathbf{J} = 0 \tag{S.6}$$

holds true by inspection of the electric field (S.2) and of the magnetic field (S.3):

$$\nabla\times\mathbf{B} \;=\; \frac{D}{2c^2}\left(\hat{\mathbf{x}}\frac{\partial}{\partial x}+\hat{\mathbf{y}}\frac{\partial}{\partial y}\right)\times(-x\hat{\mathbf{y}}+y\hat{\mathbf{x}}) \;=\; \frac{D}{2c^2}\big(-\hat{\mathbf{z}}\,-\,\hat{\mathbf{z}}\big) \;=\; -\frac{D}{c^2}\,\hat{\mathbf{z}} \tag{S.7}$$

while

$$\frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t} \;=\; -\frac{D}{c^2}\,\hat{\mathbf{z}}, \tag{S.8}$$

thus indeed

$$\nabla\times\mathbf{B} \;=\; \frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t}. \tag{S.9}$$

Problem **2**:

As explained in my notes on Liénard–Wiechert potentials, eq. (138) on page 26 and eq. (146) on page 28, the EM fields of a moving point charge obtain as

$$\mathbf{E} \;=\; \frac{Q}{4\pi\epsilon_0}\left[\frac{(1-\beta^2)(\mathbf{n}-\boldsymbol{\beta})}{\mathcal{R}^2(1-\boldsymbol{\beta}\cdot\mathbf{n})^3}\,+\,\frac{\mathbf{n}\times((\mathbf{n}-\boldsymbol{\beta})\times\mathbf{a})}{c^2\mathcal{R}(1-\boldsymbol{\beta}\cdot\mathbf{n})^3}\right]_{\rm ret} \tag{S.10}$$

where 'ret' means that everything inside the big brackets is taken at the retarded time $t_{\rm ret}=t-\mathcal{R}/c$, and

$$\mathbf{B} \;=\; \frac{\mathbf{n}^{\rm ret}}{c}\times\mathbf{E}. \tag{S.11}$$

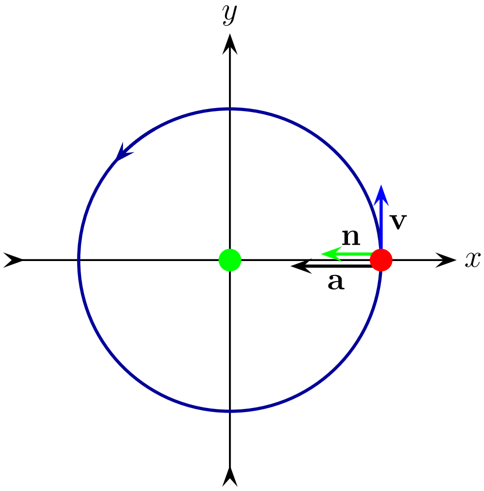
In the coordinate system shown on diagram (2)

$$\mathbf{r}(t) \;=\; b\cos(\omega t+\phi_0)\hat{\mathbf{x}} \;+\; b\sin(\omega t+\phi_0)\hat{\mathbf{y}} \quad\text{for}\quad \omega \;=\; \frac{v}{b}\,, \tag{S.12}$$

we chose the moment t such that at the retarded time $t_{\rm ret}=t-(b/c)$

$$\omega t_{\rm ret}\,+\,\phi_0 \;=\; 0 \text{ or } 2\pi\times\text{an integer} \quad\Longrightarrow\quad \mathbf{r}(t_{\rm ret}) \;=\; b\hat{\mathbf{x}}. \tag{S.13}$$

At that retarded time,



$$\begin{aligned} \mathbf{n} &= -\hat{\mathbf{x}} \\ \boldsymbol{\beta} &= \frac{v}{c}\,\hat{\mathbf{y}}, \\ \mathbf{a} &= -\frac{v^2}{b}\,\hat{\mathbf{x}} \;=\; -\frac{\beta^2 c^2}{b}\,\hat{\mathbf{x}}, \end{aligned} \tag{S.14}$$

hence

$$\begin{aligned} (1-\boldsymbol{\beta}\cdot\mathbf{n}) &= 1, \\ (\mathbf{n}-\boldsymbol{\beta}) &= -\hat{\mathbf{x}} \;-\; \beta\hat{\mathbf{y}}, \\ \frac{(1-\beta^2)(\mathbf{n}-\boldsymbol{\beta})}{\mathcal{R}^2(1-\boldsymbol{\beta}\cdot\mathbf{n})^3} &= \frac{\beta^2-1}{b^2}\,(\hat{\mathbf{x}}+\beta\hat{\mathbf{y}}), \end{aligned} \tag{S.15}$$

while

$$\begin{aligned} (\mathbf{n}-\boldsymbol{\beta})\times\frac{\mathbf{a}}{c^2} &= +\frac{\beta^3}{b}\,(\hat{\mathbf{y}}\times\hat{\mathbf{x}}) \;=\; -\frac{\beta^3}{b}\,\hat{\mathbf{z}}, \\ \frac{\mathbf{n}\times((\mathbf{n}-\boldsymbol{\beta})\times\mathbf{a})}{c^2\mathcal{R}(1-\boldsymbol{\beta}\cdot\mathbf{n})^3} &= \left(-\frac{\hat{\mathbf{x}}}{b}\right)\times\left(-\frac{\beta^3}{b}\,\hat{\mathbf{z}}\right) \\ &= +\frac{\beta^3}{b^2}\,(\hat{\mathbf{x}}\times\hat{\mathbf{z}}) \;=\; -\frac{\beta^3}{b^2}\,\hat{\mathbf{y}}. \end{aligned} \tag{S.16}$$

Altogether, the electric field at the circle's center is

$$\begin{aligned} \mathbf{E} &= \frac{Q}{4\pi\epsilon_0}\left[\frac{\beta^2-1}{b^2}\,(\hat{\mathbf{x}}+\beta\hat{\mathbf{y}}) \;-\; \frac{\beta^3}{b^2}\,\hat{\mathbf{y}}\right] \\ &= \frac{Q}{4\pi\epsilon_0 b^2}\Big((\beta^2-1)\hat{\mathbf{x}} \;+\; (\beta^3-\beta)\hat{\mathbf{y}} \;-\; \beta^3\hat{\mathbf{y}}\Big) \\ &= -\frac{Q}{4\pi\epsilon_0 b^2}\Big((1-\beta^2)\hat{\mathbf{x}} \;+\; \beta\hat{\mathbf{y}}\Big). \end{aligned} \tag{S.17}$$

Finally, the magnetic field at the same center is

$$\mathbf{B} \;=\; \frac{\mathbf{n}^{\text{ret}}}{c}\times\mathbf{E} \;=\; \frac{Q}{4\pi\epsilon_0 cb^2}\,\hat{\mathbf{x}}\times\bigl((1-\beta^2)\hat{\mathbf{x}}+\beta\hat{\mathbf{y}}\bigr)\;\frac{Q}{4\pi\epsilon_0 cb^2}\,\bigl(0+\beta(\hat{\mathbf{x}}\times\hat{\mathbf{y}})\bigr) \;=\; \frac{Q\beta}{4\pi\epsilon_0 cb^2}\,\hat{\mathbf{z}} \tag{S.18}$$

or

$$\mathbf{B} \;=\; \frac{Qv\mu_0}{4\pi b^2}\,\hat{\mathbf{z}} \tag{S.19}$$

PS: The EM fields (S.17) and (S.19) obtain at the very specific point in time, namely $t_0 = t_{\text{ret}} + b/c$ when $\omega t_{\text{ret}} + \phi_0 = 0$. To obtain the fields at other times, we should rotate them through angle $\Delta\phi = \omega(t - t_0)$. Thus:

$$\mathbf{E}(t) \;=\; -\frac{Q}{4\pi\epsilon_0 b^2}\begin{pmatrix} (1-\beta^2)\bigl(\cos(\omega(t-t_0))\hat{\mathbf{x}} \,+\, \sin(\omega(t-t_0))\hat{\mathbf{y}}\bigr) \\ +\beta\bigl(\cos(\omega(t-t_0))\,\hat{\mathbf{y}} \,-\, \sin(\omega(t-t_0))\hat{\mathbf{x}}\bigr) \end{pmatrix}, \tag{S.20}$$

while

$$\mathbf{B}(t) \;=\; \frac{Qv\mu_0}{4\pi b^2}\,\hat{\mathbf{z}} \quad \text{at all times } t. \tag{S.21}$$

Problem **3**:

Let $I(t)$ be the current through the coil. By itself, the coil has magnetic moment

$$m \;=\; NA\times I \tag{S.22}$$

but the ferrite core amplifies this magnetic moment μ-fold, thus

$$m \;=\; \mu\times NA\times I.$$

For the harmonic current

$$I(t) \;=\; I_0\cos\omega t, \tag{S.23}$$

the magnetic moment has second time derivative

$$\ddot{m} \;=\; \frac{d^2}{dt^2}\Bigl(\mu NAI_0\times\cos\omega t\Bigr) \;=\; \mu NAI_0\times(-\omega^2)\cos\omega t. \tag{S.24}$$

As explained in my notes on dipole radiation, eq. (55), this dipole radiates EM waves at the net power

$$P = \frac{\mu_0(\ddot{m})^2}{6\pi c^3} = \frac{Z_0(\ddot{m})^2}{6\pi c^4}\,, \tag{S.25}$$

thus for the antenna in question

$$P(t) = \frac{Z_0}{6\pi} \times \left(\frac{\mu N A I_0}{c^2}\right)^2 \times \cos^2 \omega t. \tag{S.26}$$

Averaging this power over the oscillation period, we have $\left\langle \cos^2(\omega t)\right\rangle = \frac{1}{2}$, hence

$$\langle P\rangle = \frac{Z_0 I_0^2}{12\pi} \times \left(\frac{\mu N A \omega^2}{c^2}\right)^2. \tag{S.27}$$

For the antenna in question

$$\begin{aligned} \frac{\mu N A \omega^2}{c^2} &= 1500 \times 60 \times (1\ \mathrm{cm}^2 = 10^{-4}\ \mathrm{m}^2) \times \left(\frac{2\pi \times 27\mathrm{MHz}}{3 \cdot 10^8\ \mathrm{m/s}}\right)^2 \\ &= 9.0\ \mathrm{m}^2 \times \left(0.566\ \mathrm{m}^{-1}\right)^2 = 2.88, \end{aligned} \tag{S.28}$$

hence

$$\frac{\langle P\rangle}{I_0^2} \approx \frac{Z_0}{12\pi} \times (2.88)^2 \approx 83\ \Omega. \tag{S.29}$$

Consequently, to radiate at $\langle P\rangle = 4$ W, the antenna should be powered by the current of amplitude

$$I_0 \approx \sqrt{\frac{4\ \mathrm{W}}{83\ \Omega}} \approx 0.22\ \mathrm{A}. \tag{S.30}$$

Problem **4**:

For a non-relativistic electron in a uniform magnetic field, the angular velocity of its circular motion does not depend on the velocity. Indeed,

$$a_c \;=\; \omega\times v \;=\; \frac{F}{m_e} \;=\; \frac{e}{m_e}\,v\times B\,, \tag{S.31}$$

hence

$$\omega \;=\; \frac{e}{m_e}\,B. \tag{S.32}$$

Consequently, the circle's radius and the acceleration are related to the electron's velocity v as

$$R \;=\; \frac{v}{\omega} \;=\; \frac{m_e v}{eB}\,, \qquad a \;=\; \omega\times v \;=\; \frac{eBv}{m_e}\,. \tag{S.33}$$

The accelerating electron radiates EM waves at the net power

$$P \;=\; \frac{e^2\mu_0}{6\pi c}\times a^2 \;=\; \frac{e^4B^2v^2\mu_0}{6\pi c m_e^2}\,. \tag{S.34}$$

The radiated power comes at the expense of the electron's kinetic energy, thus

$$\frac{d}{dt}\left(\frac{m_e v^2}{2}\right) \;=\; -P \;=\; -\frac{e^4B^2v^2\mu_0}{6\pi c m_e^2} \tag{S.35}$$

hence

$$m_e v\frac{dv}{dt} \;=\; -\frac{e^4B^2v^2\mu_0}{6\pi c m_e^2} \tag{S.36}$$

and therefore

$$\frac{dv}{dt} \;=\; -v\times\frac{e^4B^2\mu_0}{6\pi c m_e^3}\,. \tag{S.37}$$

Solving this differential equation, we immediately obtain

$$v(t) \;=\; v_0\times\exp(-t/\tau) \quad \text{for} \quad \tau \;=\; \frac{6\pi c m_e^3}{e^4\mu_0 B^2}\,. \tag{S.38}$$

Finally, in light of eq. (S.33), the orbital radius is proportional to the velocity with a constant coefficient, hence

$$R(t) \;=\; R_0 \times \exp(-t/\tau) \tag{3}$$

for exactly the same time constant τ as in eq.(S.38). Note that this time constant depends on the magnetic field as $\tau = \text{const}/B^2$.

($\star$) Numerically,

$$\frac{6\pi c m_e^3}{\mu_0 e^4} \;=\; \frac{6\pi \times (3.00\cdot 10^8\ \text{m/s}) \times (0.911\ \text{kg})^3}{(4\pi\cdot 10^{-7}\ \text{Tm/A}) \times (1.60\cdot 10^{-19}\ \text{C})^4} \;\approx\; 5.16\ \text{s}\cdot\text{T}^2, \tag{S.39}$$

so in the magnetic field $B = 1.00$ Tesla,

$$\tau \;=\; \frac{5.16\ \text{s}\cdot\text{T}^2}{B^2} \;=\; 5.16\ \text{s}. \tag{S.40}$$