Problem **1**:

(a) My coordinates: x axis along the cable, y axis horizontally across the cable, z axis vertically from one flat wire to the other wire. Since $h \ll w \ll$ cable's length L, the electric field is similar to the capacitor with two parallel $w \times L$ plates at distance h, thus

$$\mathbf{E} \;=\; \frac{V}{h}\,(-\hat{\mathbf{z}}) \tag{S.1}$$

inside the cable and negligible outside it. As to the magnetic field, it's also negligible outside the cable, while inside the cable it obtains from the Ampere Law: Treating each flat wire as a current sheet of current density $\mathbf{K} = \pm(I/w)\hat{\mathbf{x}}$, we get

$$\hat{H} \;=\; \frac{I}{w}\,\hat{\mathbf{y}}\,. \tag{S.2}$$

(b) The Poynting vector is

$$\mathbf{S} \;=\; \mathbf{E}\times\mathbf{H} \;=\; \frac{VI}{wh}(-\hat{\mathbf{z}}\times\hat{\mathbf{y}}) \;=\; \frac{VI}{wh}\,\hat{\mathbf{x}}. \tag{S.3}$$

Note the $+\hat{\mathbf{x}}$ direction of this vector — along the cable. Physically, this means the EM power flows along the cable at uniform density VI/wh, so the net power carried by the EM fields inside the cable is

$$P \;=\; S_x \times \begin{pmatrix}\text{crosssection}\\ \text{area}\end{pmatrix} \;=\; \frac{VI}{wh}\times wh \;=\; VI. \tag{S.4}$$

(c) By inspection, the net power flow $P = VI$ by the EM fields inside the cable is precisely the net electric power carried by the cable, from the source at one end of the cable to the load at the other end. But please note: this is not a separate type of EM power in addition to the ordinary electric power but simply a different way to calculate the same power $P = VI$

Problem **2**:

(a) The magnetic field of a long solenoid is concentrated inside the solenoid itself, so the magnetic flux through the wire ring is the same as the net flux through the solenoid,

$$F \;=\; \pi a^2 \times B^z_{\text{inside}} \;=\; \pi a^2 \times \mu_0(N/\ell) \times I. \tag{S.5}$$

When this flux changes with time, it induces EMF in the ring

$$\mathcal{E} \;=\; -\frac{dF}{dt} \;=\; -\pi a^2 \mu_0(N/\ell) \times \frac{dI}{dt}\,, \tag{S.6}$$

which in turn leads to the ring current

$$I_r(t) \;=\; \frac{\mathcal{E}(t)}{R} \;=\; -\frac{\pi a^2 \mu_0(N/\ell)}{R} \times \frac{dI}{dt}\,. \tag{S.7}$$

(b) Time-dependent magnetic field inside the solenoid induces circular electric field both inside and outside the solenoid. Outside the solenoid, this field is

$$\mathbf{E} \;=\; -\frac{\hat{\phi}}{2\pi s}\frac{dF}{dt} \;=\; -\pi a^2 \mu_0(N/\ell)\,\frac{dI}{dt}\,\frac{\hat{\phi}}{2\pi s} \;=\; +RI_r\,\frac{\hat{\phi}}{2\pi s} \tag{S.8}$$

In particularly, just outside the solenoid — *i.e.*, at $s = a +$ very small,

$$\mathbf{E}^{\text{outside}} \;\approx\; \frac{RI_r}{2\pi a}\,\hat{\phi}. \tag{S.9}$$

As to the magnetic field due to the current I_r in the wire ring, we know how to calculate it at the ring's center or along the ring's axis, but we do not know how to calculate it anywhere else. Fortunately, for the ring of radius $b \gg a$, we may approximate the ring's field just outside the solenoid by the field on the solenoid's axis, thus

$$\mathbf{H}^{\text{just}}_{\text{outside}} \;\approx\; \mathbf{H}_{\text{on axis}} \;=\; \frac{I_r \mu_0}{2}\,\frac{b^2 \hat{\mathbf{z}}}{(b^2+z^2)^{3/2}} \tag{S.10}$$

where z coordinate along the ring's axis is counted from the ring's center.

Finally, the Poynting vector just outside the solenoid obtains as

$$\mathbf{S}^{\rm just}_{\rm outside} \;=\; \mathbf{E}^{\rm just}_{\rm outside}\times\mathbf{H}^{\rm just}_{\rm outside} \;=\; \frac{RI_r^2}{4\pi a}\,\frac{b^2}{(b^2+z^2)^{3/2}}\,\bigl(\hat{\phi}\times\hat{\mathbf{z}}\;=\;\hat{\mathbf{s}}\bigr). \tag{S.11}$$

Note the direction of this vector: the EM power flows in the radial direction away from the solenoid.

(c) The net EM power flowing away from the solenoid obtains by integration

$$W_{\rm net} \;=\; \iint \mathbf{S}\cdot\mathbf{d^2a} \tag{S.12}$$

over a cylindrical surface immediately surrounding the solenoid. For such a surface $\mathbf{d^2a} = a\,d\phi\,dz\,\hat{\mathbf{s}}$, hence

$$\mathbf{S}\cdot\mathbf{d^2a} \;=\; aS^s\,d\phi\,dz \;\to\; 2\pi a\,S^s\,dz, \tag{S.13}$$

and therefore

$$\begin{aligned} W_{\rm net} &= \int\limits_{-\infty}^{+\infty}\frac{RI_r^2}{4\pi a}\times\frac{b^2}{(b^2+z^2)^{3/2}}\times 2\pi a\,dz \\ &= \frac{RI_r^2}{2}\int\limits_{-\infty}^{+\infty}\frac{b^2\,dz}{(b^2+z^2)^{3/2}} \\ &= \frac{RI_r^2}{2}\times 2 \;=\; RI_r^2\,. \end{aligned} \tag{S.14}$$

By inspection, this is precisely the electric power dissipated by the wire ring, *quod erat demonstrandum.*

Problem **3**:

(a) Let the two charges be located at $\mathbf{r}_{1,2} = (0,0,\pm a)$, so the equidistant plane is the (x,y)

plane. Along that plane, the net electric field is

$$\mathbf{E} \;=\; \frac{Q}{4\pi\epsilon_0}\,\frac{(x,y,-a)}{R^3} \;+\; \frac{Q}{4\pi\epsilon_0}\,\frac{(x,y,+a)}{R^3} \;=\; \frac{Q}{2\pi\epsilon_0}\,\frac{(x,y,0)}{R^3} \tag{S.15}$$

where $R=\sqrt{x^2+y^2+a^2}$. Consequently, the electric stress tensor is

$$T^{ij} \;=\; \epsilon_0 E^i E^j \;-\; \tfrac{1}{2}\epsilon_0 \mathbf{E}^2 \delta^{ij}\,, \tag{S.16}$$

which in matrix notations becomes

$$\begin{aligned}\overleftrightarrow{T} \;&=\; \frac{Q^2}{4\pi^2\epsilon_0\,R^6}\begin{pmatrix} x^2 & xy & 0\\ xy & y^2 & 0\\ 0 & 0 & 0\end{pmatrix} \;-\; \frac{Q^2(x^2+y^2)}{8\pi^2\epsilon_0\,R^6}\begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\\ &=\; \frac{Q^2}{8\pi^2\epsilon_0\,R^6}\begin{pmatrix} x^2-y^2 & 2xy & 0\\ 2xy & y^2-x^2 & 0\\ 0 & 0 & -x^2-y^2\end{pmatrix}.\end{aligned} \tag{S.17}$$

(b) This time, the electric field on the equidistant (x,y) plane is

$$\mathbf{E} \;=\; \frac{Q}{4\pi\epsilon_0}\,\frac{(x,y,-a)}{R^3} \;-\; \frac{Q}{4\pi\epsilon_0}\,\frac{(x,y,+a)}{R^3} \;=\; \frac{Q}{2\pi\epsilon_0}\,\frac{(0,0,a)}{R^3}\,. \tag{S.18}$$

Consequently, the electric stress tensor

$$T^{ij} \;=\; \epsilon_0 E^i E^j \;-\; \tfrac{1}{2}\epsilon_0 \mathbf{E}^2 \delta^{ij} \tag{S.19}$$

becomes in matrix notations

$$\begin{aligned}\overleftrightarrow{T} \;&=\; \frac{Q^2 a^2}{4\pi^2\epsilon_0\,R^6}\begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{pmatrix} \;-\; \frac{Q^2 a^2}{8\pi^2\epsilon_0\,R^6}\begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\\ &=\; \frac{Q^2 a^2}{8\pi^2\epsilon_0\,R^6}\begin{pmatrix} -1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & +1\end{pmatrix}.\end{aligned} \tag{S.20}$$

Problem **4**:

Along the equatorial plane — both inside and outside the sphere — the magnetic field is parallel to the rotation axis, which we take to be the z axis of our coordinate system:

$$\mathbf{B}_{\text{inside}} = +\frac{2\mu_0\sigma R\omega}{3}\,\hat{\mathbf{z}}, \qquad \mathbf{B}_{\text{outside}} = -\frac{\mu_0\sigma R^4\omega}{3r^3}\,\hat{\mathbf{z}}. \tag{S.21}$$

Consequently, in the equatorial plane

$$\overleftrightarrow{T}_{\text{mag}} = \frac{B(r)^2}{2\mu_0}\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & +1 \end{pmatrix}. \tag{S.22}$$

Given this stress tensor, the net (magnetic) force between the two hemispheres obtains as

$$\mathbf{F} = -\iint \overleftrightarrow{T}_{\text{mag}}\cdot\mathbf{d^2a} \tag{S.23}$$

where the integral is over the entire equatorial plane, both inside and inside the sphere in question. Along the whole plane $\mathbf{d^2a}$ points in the z direction, and the net force should also point in the same direction because of rotational symmetry. Therefore, $F^x = F^y = 0$ while

$$F^z = -\iint T^{zz}\,d^2a = -\int_0^\infty \frac{B(r)^2}{2\mu_0}\,2\pi r\,dr. \tag{S.24}$$

where the overall $-$ sign indicates the attractive force between the two hemispheres. Specifically,

$$\frac{B^2(r)}{2\mu_0} = \frac{\mu_0(\sigma R\omega)^2}{18}\times\begin{cases} 4 & \text{for } r < R, \\ (R/r)^6 & \text{for } r > R, \end{cases} \tag{S.25}$$

hence

$$F^z = -\frac{2\pi\mu_0(\sigma R\omega)^2}{18}\times\left(\int_0^R 4\times r\,dr + \int_R^\infty \frac{R^6}{r^6}\,r\,dr\right). \tag{S.26}$$

Evaluating the integrals, we get

$$\int_0^R 4\times r\,dr = 4\times\frac{R^2}{2} = 2R^2, \tag{S.27}$$

$$\int_R^\infty \frac{R^6}{r^6}\, r\,dr \;=\; R^6 \times \frac{1}{4R^4} \;=\; \tfrac{1}{4}R^2, \tag{S.28}$$

hence

$$F^z \;=\; -\frac{2\pi\mu_0(\sigma R\omega)^2}{18} \times \left(2R^2 \,+\, \tfrac{1}{4}R^2 \;=\; \tfrac{9}{4}R^2\right) \;=\; -\frac{\pi\mu_0\sigma^2 R^4\omega^2}{4}\,. \tag{S.29}$$

Problem **5**:

(a) Let's assume the top plate of the capacitor has positive charge $Q > 0$. Then the electric displacement field between the capacitor plates

$$\mathbf{D} \;=\; -(Q/A)\hat{\mathbf{z}} \tag{S.30}$$

points down, while the magnetic field is the external field $\mathbf{B} = b\hat{\mathbf{x}}$. Consequently, the EM momentum density between the plates

$$\mathbf{g} \;=\; \mathbf{D}\times\mathbf{B} \;=\; (Q/A)B\big(-\hat{\mathbf{z}}\times\hat{\mathbf{x}} \;=\; -\hat{\mathbf{y}}\big) \tag{S.31}$$

is directed along the negative y axis, and the net EM momentum of the capacitor is simply the volume times this momentum density,

$$\mathbf{p}_{\rm em} \;=\; Ah\,\mathbf{g} \;=\; -QBh\,\hat{\mathbf{y}}\,. \tag{S.32}$$

(b) Assuming the positive charge $Q > 0$ is on the top plate of the capacitor, the discharging current in the wire lows downward. Consequently, the magnetic force on the wire

$$\mathbf{F} \;=\; I\vec{\ell}\times B \;=\; Ih(-\hat{\mathbf{z}})\times B\hat{\mathbf{x}} \;=\; -IhB\,\hat{\mathbf{y}} \tag{S.33}$$

also points in the negative y direction. And the net impulse of this magnetic force is

$$\vec{\Pi} \;=\; \int\mathbf{F}\,dt \;=\; -hB\,\hat{\mathbf{y}}\int I(t)\,dt \;=\; -hB\,\hat{\mathbf{y}}\,Q. \tag{S.34}$$

(c) By inspection, the net impulse of the magnetic force is precisely the net EM momentum of the capacitor before its discharge,

$$\vec{\Pi} \;=\; -QhB\,\hat{\mathbf{y}} \;=\; +\mathbf{p}_{\text{em}}\,. \tag{S.35}$$

This is in perfect agreement with the momentum-impulse theorem:

$$\mathbf{p}_{\text{init}}^{\text{system}} \;-\; \mathbf{p}_{\text{final}}^{\text{system}} \;=\; \vec{\Pi}\binom{\text{by the system}}{\text{on other bodies}}. \tag{S.36}$$

The system in this problem is the EM field inside the capacitor — whose net momentum drops from $-QhB\hat{\mathbf{y}}$ to zero, — while the other bodies is simply the wire discharging the capacitor.

<u>Problem **6**</u>:

(a) Inside the toroidal coil, there is magnetic field

$$\mathbf{B} \;=\; \frac{\mu_0 NI}{2\pi s}\,\hat{\phi} \;\approx\; \frac{\mu_0 NI}{2\pi a}\,\hat{\phi}\,. \tag{S.37}$$

Also, the point charge at the toroid's center generates the electric field

$$\mathbf{E} \;=\; \frac{Q}{4\pi\epsilon_0}\,\frac{\mathbf{n}}{r^2}\,. \tag{S.38}$$

Assuming the coil's wires do not screen out this electric field, we may approximate the electric field inside the coil as

$$\mathbf{E} \;\approx\; \frac{Q}{4\pi\epsilon_0 a^2}\,\big(\mathbf{n}\approx\hat{\mathbf{s}}\big). \tag{S.39}$$

Together, the electric and the magnetic fields inside the coil have momentum density

$$\mathbf{g}_{\text{em}} \;=\; \mathbf{D}\times\mathbf{B} \;=\; \epsilon_0\mathbf{E}\times\mathbf{B} \;=\; \frac{Q}{4\pi a^2}\,\frac{\mu_0 NI}{2\pi a}\,\big(\hat{\mathbf{s}}\times\hat{\phi}\;=\;\hat{\mathbf{z}}\big) \;=\; \frac{\mu_0 NIQ}{8\pi^2 a^3}\,\hat{\mathbf{z}}\,. \tag{S.40}$$

Note: this momentum is directed along the z axis of the toroidal coil.

Finally, since this momentum density is approximately uniform inside the coil and vanishes outside the coil, the net EM momentum obtains by simply multiplying the momentum density (S.40) but the internal volume $\mathcal{V} = 2\pi a \times w \times h$ of the coil, thus

$$\mathbf{p}_{\text{em}}^{\text{net}} = 2\pi awh \frac{\mu_0 NIQ}{8\pi^2 a^3} \hat{\mathbf{z}} = \frac{\mu_0 NIQwh}{4\pi a^2} \hat{\mathbf{z}}. \tag{S.41}$$

(b) As you should have learned last semester, finding the electric field induced by changing the magnetic flux in a thin coil is mathematically similar to finding the magnetic field due to a current in a thin wire, except for an overall sign. Indeed, compare Maxwell equations

$$\nabla \times E = -\frac{\partial \mathbf{B}}{\partial t} \quad \text{and} \quad \nabla \times \mathbf{H} = +\mathbf{J} + \mathbf{J}_{\text{displ}}. \tag{S.42}$$

In particular, for a thin toroidal coil with time-dependent flux $F(t)$, the electric field induced at the coil's center is is similar to the magnetic field at the center of a current carrying coil,

$$\mathbf{H} = +I \frac{\hat{\mathbf{z}}}{2a} \quad \Longrightarrow \quad \mathbf{E} = -\frac{dF}{dt} \frac{\hat{\mathbf{z}}}{2a}. \tag{S.43}$$

For the coil in question,

$$F(t) = \frac{\mu_0 whNI(t)}{2\pi a} \tag{S.44}$$

hence

$$\mathbf{E} = -\frac{\mu_0 whN}{4\pi a^2} \frac{dI}{dt} \hat{\mathbf{z}}, \tag{S.45}$$

which exerts the force

$$\mathbf{F} = Q\mathbf{E} = -\frac{\mu_0 whNQ}{4\pi a^2} \frac{dI}{dt} \hat{\mathbf{z}} \tag{S.46}$$

on the point charge at the coil's center. The net impulse of this force is

$$\vec{\Pi} = \int \mathbf{F}\, dt = \frac{\mu_0 whNQ}{4\pi a^2} (-\hat{\mathbf{z}}) \int \frac{dI}{dt}\, dt = \frac{\mu_0 whNQ}{4\pi a^2} (-\hat{\mathbf{z}}) \big(I_{\text{fin}} - I_{\text{init}}\big). \tag{S.47}$$

For the problem in question, we start with some current I_0 and then turn it off, thus $I_{\text{fin}} - I_{\text{init}} = -I_0$, thus the net impulse of the electric force on the point charge

$$\vec{\Pi} = +\frac{\mu_0 whNQI_0}{4\pi a^2} \hat{\mathbf{z}}. \tag{S.48}$$

(c) By inspection, the net impulse (S.48) of the force on the point charge is precisely the net EM momentum in the coil before we turned off the current,

$$\vec{\Pi} \;=\; +\frac{\mu_0 whNQI_0}{4\pi a^2}\hat{\mathbf{z}} \;=\; \mathbf{p}_{\rm em}^{\rm init}\,. \tag{S.49}$$

This is in perfect agreement with the momentum-impulse theorem

$$\vec{\Pi} \;=\; \mathbf{p}_{\rm em}^{\rm init} \;-\; \mathbf{p}_{\rm em}^{\rm fin} \tag{S.50}$$

since after we turn off the current $\mathbf{p}_{\rm em}^{\rm fin} = 0$.

Problem **7**:

Inside the sphere, the EM momentum density is uniform

$$\mathbf{g} \;=\; \mathbf{D}\times\mathbf{B} \;=\; \frac{4\mu_0}{9}\,\mathbf{P}\times\mathbf{M}\,. \tag{S.51}$$

Outside the sphere, we have a more complicated formula

$$\mathbf{g} \;=\; \mathbf{D}\times\mathbf{B} \;=\; \frac{\mu_0 R^6}{9r^6}\,\big(3\mathbf{n}(\mathbf{n}\cdot\mathbf{P} \,-\, \mathbf{P}\big)\times\big(3\mathbf{n}(\mathbf{n}\cdot\mathbf{M} \,-\, \mathbf{M}\big), \tag{S.52}$$

where

$$\begin{aligned}\big(3\mathbf{n}(\mathbf{n}\cdot\mathbf{P}) \,-\, \mathbf{P}\big)\times\big(3\mathbf{n}(\mathbf{n}\cdot\mathbf{M}) \,-\, \mathbf{M}\big) \;&=\\ =\; 9(\mathbf{n}\cdot\mathbf{P})(\mathbf{n}\cdot\mathbf{M})(\mathbf{n}\times\mathbf{n}) \;&+\; \mathbf{P}\times\mathbf{M}\\ -\; 3(\mathbf{n}\cdot\mathbf{M})(\mathbf{n}\times\mathbf{P}) \;&-\; 3(\mathbf{n}\cdot\mathbf{P})(\mathbf{n}\times\mathbf{M}).\end{aligned} \tag{S.53}$$

On the RHS of this formula, the first term vanishes because $\mathbf{n}\times\mathbf{n} = 0$, while the last two terms cancel each other after averaging over the direction of $\mathbf{n}$. Note that we are going to integrate the momentum density (S.52) over the whole space outside the sphere — which includes both the radial integration and integration over the directions $\mathbf{n}$, — so we may just as well average over the directions $\mathbf{n}$ before integrating over the radius r. In light of eq. (7), angular averaging yields

$$3(\mathbf{n}\cdot\mathbf{P})(\mathbf{n}\times\mathbf{M}) \;\to\; \mathbf{P}\times\mathbf{M}\,,$$
$$3(\mathbf{n}\cdot\mathbf{M})(\mathbf{n}\times\mathbf{P}) \;\to\; \mathbf{M}\times\mathbf{P}\,, \tag{S.54}$$

hence

$$-3(\mathbf{n}\cdot\mathbf{M})(\mathbf{n}\times\mathbf{P}) \;-\; 3(\mathbf{n}\cdot\mathbf{P})(\mathbf{n}\times\mathbf{M}) \;\to\; -\mathbf{P}\times\mathbf{M} \;-\; \mathbf{M}\times\mathbf{P} \;=\; 0. \tag{S.55}$$

Consequently, the only surviving term on the RHS of eq. (S.53) is $+\mathbf{P}\times\mathbf{M}$, thus

$$\mathbf{g} \;\to\; \frac{\mu_0 R^6}{9r^6}\,(\mathbf{P}\times\mathbf{M}). \tag{S.56}$$

Altogether, after angular averaging but before radial integration we have

$$\mathbf{g} \;\to\; \frac{\mu_0}{9}(\mathbf{P}\times\mathbf{M})\begin{cases} 4 & \text{for } r > R,\\ (R/r)^6 & \text{for } r > R.\end{cases} \tag{S.57}$$

Integrating this momentum density over the radius, we have and the net EM momentum generated by the ball is the integral of this density over the whole space:

$$\mathbf{p}_{\text{em}}^{\text{net}} \;=\; \frac{\mu_0}{9}(\mathbf{P}\times\mathbf{M})\int\limits_0^\infty \left\{\begin{matrix} 4 & \text{for } r > R\\ (R/r)^6 & \text{for } r > R\end{matrix}\right\} 4\pi r^2\,dr, \tag{S.58}$$

where the integral over all the radii splits into two ranges, one for $r < R$ (inside the sphere) and one for $r > R$ (outside the sphere. Thus

$$\int\limits_0^R 4\times 4\pi r^2 dr \;=\; 16\pi\times\frac{R^3}{3} \;=\; \frac{16\pi}{3}\,R^3\,, \tag{S.59}$$

$$\int\limits_R^\infty (R/r)^6\times 4\pi r^2\,dr \;=\; 4\pi R^6\times\frac{1}{3R^3} \;=\; \frac{4\pi}{3}\,R^3\,, \tag{S.60}$$

hence

$$\begin{pmatrix}\text{whole}\\ \text{integral}\end{pmatrix} \;=\; \frac{16\pi}{3}R^3 \;+\; \frac{4\pi}{4}R^3 \;=\; \frac{20\pi}{3}R^3\,, \tag{S.61}$$

and therefore

$$\mathbf{p}_{\text{em}}^{\text{net}} \;=\; \frac{20\pi\mu_0 R^3}{27}\,(\mathbf{P}\times\mathbf{M}). \tag{S.62}$$

<u>Problem **8**</u>:

(q) Let's start with the local energy conservation eq. (15). Regardless of the dielectric/magnetic material non-uniformity

$$\begin{aligned}\frac{\partial u}{\partial t} &= \mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t} \;+\; \mathbf{H}\cdot\frac{\partial\mathbf{B}}{\partial t}\\ &= \mathbf{E}\cdot\bigl(\nabla\times\mathbf{H}-\mathbf{J}\bigr) \;+\; \mathbf{H}\cdot\bigl(-\nabla\times\mathbf{E}\bigr)\\ &= -\mathbf{E}\cdot\mathbf{J} \;+\; \mathbf{E}\cdot(\nabla\times\mathbf{H}) \;-\; \mathbf{H}\cdot(\nabla\times\mathbf{E})\\ &= -\mathbf{E}\cdot\mathbf{J} \;+\; \nabla\cdot(\mathbf{E}\times\mathbf{H})\\ &= P \;+\; \nabla\cdot\mathbf{S},\end{aligned} \tag{S.63}$$

hence eq. (15).

But the local momentum conservation becomes more complicated. Let's start with the electric part of the $\overleftrightarrow{T}$ stress tensor and its divergence. In components

$$\begin{aligned}\nabla^i T^{ij}_{\text{el}} &= \nabla^i\Bigl(D^iE^j \;-\; \tfrac{1}{2}\epsilon\epsilon_0\mathbf{E}^2\delta^{ij}\Bigr)\\ &= (\nabla^iD^i)E^j \;+\; D^i(\nabla^iE^j) \;-\; \epsilon\epsilon_0E^k\nabla^jE^k \;-\; \tfrac{1}{2}(\nabla^j\epsilon)\epsilon_0\mathbf{E}^2\end{aligned} \tag{S.64}$$

where $\nabla^iD^i = \nabla\cdot\mathbf{D} = \rho$ while

$$\begin{aligned}D^i(\nabla^iE^j) \;-\; D^k(\nabla^jE^k) &= D^i(\nabla^iE^j) \;-\; D^i(\nabla^jE^i)\\ &= D^i\bigl(\nabla^iE^j - \nabla^jE^i \;=\; \epsilon^{ijk}(\nabla\times\mathbf{E})^k\bigr)\\ &= \bigl[(\nabla\times\mathbf{E})\times\mathbf{D}\bigr]^j,\end{aligned} \tag{S.65}$$

hence in vector notations

$$\nabla\cdot\overleftrightarrow{T}_{\text{el}} \;=\; \rho\mathbf{E} \;+\; (\nabla\times\mathbf{E})\times\mathbf{D} \;-\; \tfrac{1}{2}\epsilon_0\mathbf{E}^2(\nabla\epsilon). \tag{S.66}$$

Similarly, the magnetic part of the stress tensor has divergence

$$\begin{aligned}\nabla^i T^{ij}_{\rm mag} &= \nabla^i\Bigl(B^i H^j \,-\, \tfrac{1}{2}\mu\mu_0\mathbf{H}^2\delta^{ij}\Bigr)\\ &= (\nabla^i B^i)H^j \,+\, B^i(\nabla^i H^j) \,-\, \mu\mu_0 H^k\nabla^i H^k \,-\, \tfrac{1}{2}(\nabla^i\mu)\mu_0\mathbf{H}^2\end{aligned} \tag{S.67}$$

where $\nabla^i B^i = \nabla\cdot\mathbf{B} = 0$ while

$$\begin{aligned}B^i(\nabla^i H^j) \,-\, B^k(\nabla^j H^k) &= B^i(\nabla^i H^j) \,-\, B^i(\nabla^j H^i)\\ &= B^i\bigl(\nabla^i H^j - \nabla^j H^i \,=\, \epsilon^{ijk}(\nabla\times\mathbf{H})^k\bigr)\\ &= \bigl[(\nabla\times\mathbf{H})\times\mathbf{B}\bigr]^j,\end{aligned} \tag{S.68}$$

hence in vector notations

$$\nabla\cdot\overleftrightarrow{T}_{\rm mag} \,=\, 0 \,+\, (\nabla\times\mathbf{H})\times\mathbf{B} \,-\, \tfrac{1}{2}\mu_0\mathbf{H}^2(\nabla\mu). \tag{S.69}$$

Altogether,

$$\nabla\cdot\overleftrightarrow{T}_{\rm em} \,=\, \rho\mathbf{E} \,+\, (\nabla\times\mathbf{E})\times\mathbf{D} \,-\, \tfrac{1}{2}\epsilon_0\mathbf{E}^2(\nabla\epsilon) +\, (\nabla\times\mathbf{H})\times\mathbf{B} \,-\, \tfrac{1}{2}\mu_0\mathbf{H}^2(\nabla\mu), \tag{S.70}$$

and consequently — for the force density as in eq. (18), —

$$\nabla\cdot\overleftrightarrow{T}_{\rm em} \,-\, \mathbf{f}_{\rm em} \,=\, (\nabla\times\mathbf{E})\times\mathbf{D} \,+\, (\nabla\times\mathbf{H})\times\mathbf{B} \,-\, \mathbf{J}\times\mathbf{B}. \tag{S.71}$$

But by Maxwell equations

$$\nabla\times\mathbf{E} \,=\, -\frac{\partial\mathbf{B}}{\partial t}\,, \qquad \nabla\times\mathbf{H} \,=\, +\mathbf{J} \,+\, \frac{\partial\mathbf{D}}{\partial t}\,, \tag{S.72}$$

hence eq. (S.71)becomes

$$\begin{aligned}\nabla\cdot\overleftrightarrow{T}_{\rm em} \,-\, \mathbf{f}_{\rm em} &= -\frac{\partial\mathbf{B}}{\partial t}\times\mathbf{D} \,+\, \left(\mathbf{J}+\frac{\partial\mathbf{D}}{\partial t}\right)\times\mathbf{B} \,-\, \mathbf{J}\times\mathbf{B}\\ &= +\mathbf{D}\times\frac{\partial\mathbf{B}}{\partial t} \,+\, \frac{\partial\mathbf{D}}{\partial t}\times\mathbf{B}\\ &= \frac{\partial}{\partial t}\Bigl(\mathbf{D}\times\mathbf{B}\Bigr) \,=\, \frac{\partial\mathbf{g}_{\rm em}}{\partial t}\,.\end{aligned} \tag{S.73}$$

Thus, the EM momentum density (13), the stress tensor (14), and the force density (18) indeed obey the local momentum conservation equation

$$\frac{\partial \mathbf{g}}{\partial t} \;-\; \nabla\cdot\overset{\leftrightarrow}{T} \,+\, \mathbf{f} \;=\; 0. \tag{16}$$

Quod erat demonstrandum.

(b–c) As written, the dielectric force term and the magnetic force term in eq. (18) apply to the continuously variable electric permittivity $\epsilon(\mathbf{r})$ and magnetic permeability $\mu(r)$. For abrupt discontinuities of $\epsilon(\mathbf{r})$ and/or $\mu(\mathbf{r})$ at the boundaries of dielectric or magnetic materials, we need to include the delta-functions

$$\nabla\epsilon(\mathbf{r}) \;\to\; \mathrm{disc}(\epsilon)\delta(x_{\perp\mathrm{boundary}})\mathbf{n}_{\perp\mathrm{boundary}} \tag{S.74}$$

and likewise

$$\nabla\mu(\mathbf{r}) \;\to\; \mathrm{disc}(\mu)\delta(x_{\perp\mathrm{boundary}})\mathbf{n}_{\perp\mathrm{boundary}}\,. \tag{S.75}$$

In particular, for a piece of uniform dielectric (with $\epsilon\equiv\epsilon_d$) surrounded by vacuum (with $\epsilon\equiv 1$), we have $\mathrm{disc}(\epsilon)=(1-\epsilon_d)$, thus

$$\nabla\epsilon(\mathbf{r}) \;=\; (1-\epsilon_d)\delta(x_{\perp\mathrm{boundary}})\mathbf{n}_{\perp\mathrm{boundary}}, \tag{S.76}$$

and hence in the integral for the net force on the dielectric

$$(\nabla\epsilon)\,d^3\mathrm{Vol} \;\to\; (1-\epsilon_d)\,\mathbf{d^2a}[\text{over the boundary of the dielectric piece}]. \tag{S.77}$$

Thus, for a dielectric piece occupying volume $\mathcal{V}$ and not carrying any free charges, the net electric force is

$$\begin{aligned}
\mathbf{F}^{\mathrm{net}}_{\mathrm{el}} &= \iiint\bigl(-\tfrac{1}{2}\epsilon_0\mathbf{E}^2\bigr)(\nabla\epsilon)d^3\mathrm{Vol} \\
&= \oiint\limits_{\substack{\text{boundary}\\ \text{of } \mathcal{V}}} \tfrac{1}{2}\epsilon_0(\epsilon_d-1)\mathbf{E}^2\,\mathbf{d^2a} \\
&\qquad \langle\!\langle \text{ by Gauss theorem } \rangle\!\rangle \\
&= \tfrac{1}{2}\epsilon_0(\epsilon_d-1)\iiint\limits_{\mathcal{V}} \nabla(\mathbf{E}^2)\,d^3\mathrm{Vol}.
\end{aligned} \tag{S.78}$$

In particular, when this dielectric piece is small enough that the gradient $\nabla(\mathbf{E}^2)$ is approximately uniform over its volume $\mathcal{V}$,

$$\nabla(\mathbf{E}^2)@(\mathbf{r}\in\mathcal{V}) \;\approx\; \nabla(\mathbf{E}^2)@\mathbf{r}_d \tag{S.79}$$

for some representative point $\mathbf{r}_d$Z inside the dielectric piece, the integral (S.78) simplifies to

$$\mathbf{F}^{\rm net}_{\rm el} \;=\; \tfrac{1}{2}\epsilon_0(\epsilon_d-1)\mathcal{V}\nabla(\mathbf{E}^2)@\mathbf{r}_d\,. \tag{19}$$

Likewise, for a piece of a uniform magnetic material with some $\mu_m\neq 1$ surrounded by vacuum,

$$\nabla\mu(\mathbf{r}) \;=\; (1-\mu_m)\delta(x_{\perp\rm boundary})\mathbf{n}_{\perp\rm boundary}, \tag{S.80}$$

and hence in the integral for the net force on the material

$$(\nabla\mu)\,d^3{\rm Vol} \;\to\; (1-\mu_m)\,\mathbf{d^2a}[\text{over the boundary of the dielectric piece}]. \tag{S.81}$$

Consequently, the net magnetic force is

$$\begin{aligned}\mathbf{F}^{\rm net}_{\rm mag} &= \iiint\bigl(-\tfrac{1}{2}\mu_0\mathbf{H}^2\bigr)(\nabla\mu)d^3{\rm Vol}\\ &= \oiint\limits_{\substack{\text{boundary}\\ \text{of }\mathcal{V}}} \tfrac{1}{2}\mu_0(\mu_m-1)\mathbf{H}^2\,\mathbf{d^2a}\\ &\qquad\langle\!\langle\text{ by Gauss theorem }\rangle\!\rangle\\ &= \tfrac{1}{2}\epsilon_0(\mu_m-1)\iiint\limits_{\mathcal{V}}\nabla(\mathbf{H}^2)\,d^3{\rm Vol},\end{aligned} \tag{S.82}$$

which for a small piece of magnetic material may be approximated as

$$\mathbf{F}^{\rm net}_{\rm mag} \;=\; \tfrac{1}{2}\mu_0(\mu_m-1)\mathcal{V}\nabla(\mathbf{H}^2)@\mathbf{r}_m\,. \tag{S.83}$$

Note: for a paramagnetic or ferromagnetic material with $\mu_m>1$, this force points in the direction of increasing magnetic field, wile for a diamagnetic material with $\mu_m<1$ the force has opposite direction. OOH, the dielectric force (19) always points in the direction of increasing electric field since all dielectrics have $\epsilon_d>1$.