Problem **1**:

(a) The EM angular momentum density is

$$\vec{\ell} = \mathbf{r}\times\mathbf{g} = \mathbf{r}\times(\mathbf{D}\times\mathbf{B}). \tag{S.1}$$

For the system in question, the magnetic field $\mathbf{B} = B\hat{\mathbf{z}}$ is uniform throughout the system but the electric displacement field

$$\mathbf{D} = \frac{Q}{4\pi r^2}\mathbf{n} \tag{S.2}$$

exists only in the space between the two spheres, at $a < r < b$. In that space

$$\mathbf{g} = \frac{Q}{4\pi r^2}\mathbf{n}\times B\hat{\mathbf{z}} = \frac{QB}{4\pi r^2}\left(-\sin\theta\hat{\boldsymbol{\phi}}\right), \tag{S.3}$$

which rotates clockwise (around z axis) for $QB > 0$ or counterclockwise for $QB < 0$. Either way, the net angular momentum points in the $\pm z$ direction, so all we need is the z component of its density,

$$\ell_z = r\sin\theta\times g_\phi = r\sin\theta\times\frac{-QB\sin\theta}{4\pi r^2} = -\frac{QB\sin^2\theta}{4\pi r}. \tag{S.4}$$

It remain to integrate this density over the space between the two spheres,

$$\begin{aligned} L_z &= \iiint d^3\mathrm{Vol}\,\ell_z \\ &= \int_a^b dr\,r^2\times\int_0^\pi d\theta\,2\pi\sin\theta\times\frac{-QB\sin^2\theta}{4\pi r} \\ &= -\frac{QB}{2}\times\int_a^b dr\,r\times\int_0^\pi d\theta\,\sin^3\theta \\ &= -\frac{QB}{2}\times\frac{b^2-a^2}{2}\times\frac{4}{3} \\ &= -\tfrac{1}{3}QB(b^2-a^2)\,. \end{aligned} \tag{S.5}$$

This is the net EM angular momentum of the before we turn off the magnetic field.

(b) A uniform but time-dependent magnetic field $\mathbf{B}(t) = B(t)\hat{\mathbf{z}}$ induces electric field in the circular direction around the z axis,

$$\mathbf{E} \;=\; -\frac{1}{2}\frac{dB}{dt}\,r\sin\theta\hat{\boldsymbol{\phi}}\,. \tag{S.6}$$

This field exerts torques on the two charged spherical shells in the directions $\pm\hat{\mathbf{z}}$. For each shell,

$$\frac{d\tau_z}{da} \;=\; r\sin\theta\times\sigma E_\phi \;=\; -\frac{1}{2}\frac{dB}{dt}\times\frac{\text{charge}}{4\pi r^2}\times r^2\sin^2\theta \;=\; \mp\frac{Q}{8\pi}\frac{dB}{dt}\sin^2\theta. \tag{S.7}$$

Integrating this torque density over the area of each shell, we have

$$\begin{aligned}\tau_z^{\text{net}}\binom{\text{inner}}{\text{shell}} \;&=\; -\frac{Q}{8\pi}\frac{dB}{dt}\times\iint\limits_{\text{inner}}\sin^2\theta\,d^2a\\ &=\; -\frac{Q}{8\pi}\frac{dB}{dt}\times\frac{8\pi a^2}{3}\\ &=\; -\frac{Qa^2}{3}\times\frac{dB}{dt}\,,\end{aligned} \tag{S.8}$$

and likewise

$$\begin{aligned}\tau_z^{\text{net}}\binom{\text{outer}}{\text{shell}} \;&=\; +\frac{Q}{8\pi}\frac{dB}{dt}\times\iint\limits_{\text{outer}}\sin^2\theta\,d^2a\\ &=\; +\frac{Q}{8\pi}\frac{dB}{dt}\times\frac{8\pi b^2}{3}\\ &=\; +\frac{Qb^2}{3}\times\frac{dB}{dt}\,.\end{aligned} \tag{S.9}$$

Altogether, the net torque on the two-shell system is

$$\tau_z^{\text{net}} \;=\; \tfrac{1}{3}Q(b^2-a^2)\,\frac{dB}{dt}\,. \tag{S.10}$$

Consequently, as we change the B field, this torque imparts mechanical angular momentum

$$\Delta L_z \;=\; \tfrac{1}{3}Q(b^2-a^2)\times\int\frac{dB}{dt}\,dt \;=\; \tfrac{1}{3}Q(b^2-a^2)\times\big(B_{\text{fin}}-B_{\text{init}}\big). \tag{S.11}$$

In particular, if we start with some magnetic field $B_{\text{init}} = B$ and then turn it off completely,

$B_{\rm fin} = 0$, then

$$\Delta L_z \;=\; -\tfrac{1}{3}Q(b^2 - a^2) \times B. \tag{S.12}$$

By inspection, this is precisely the initial angular momentum of the EM fields, *cf.* part (a).

<u>Problem **2**</u>:

(a) Inside the iron ball, there is no electric field while the magnetic field is uniform $\mathbf{B} = \frac{2}{3}\mu_0\mathbf{M}$. Outside the ball, there is radial electric field

$$\mathbf{D} \;=\; \frac{Q}{4\pi r^2}\mathbf{n} \tag{S.13}$$

and a dipole magnetic field

$$\mathbf{B} \;=\; \frac{\mu_0}{4\pi}\,\frac{3(\mathbf{n}\cdot\mathbf{m})\mathbf{n} \,-\, \mathbf{m}}{r^3} \tag{S.14}$$

where

$$\mathbf{m} \;=\; \frac{4\pi}{3}R^3\,\mathbf{M} \tag{S.15}$$

is the net magnetic dipole moment of the ball. Consequently, the EM momentum density $\mathbf{g} = \mathbf{D}\times\mathbf{B}$ vanishes inside the ball, while outside the ball

$$\mathbf{g} \;=\; \frac{Q\mu_0}{16\pi^2}\,\frac{\mathbf{n}\times\big(3(\mathbf{n}\cdot\mathbf{m})\mathbf{n} - \mathbf{m}\big)}{r^5} \;=\; -\frac{Q\mu_0}{16\pi^2}\,\frac{\mathbf{n}\times\mathbf{m}}{r^3} \;=\; -\frac{QMR^3\mu_0}{12\pi}\,\frac{\mathbf{n}\times\hat{\mathbf{z}}}{r^5}\,. \tag{S.16}$$

In spherical coordinates $\mathbf{n}\times\hat{\mathbf{z}} = -\sin\theta\hat{\boldsymbol{\phi}}$, hence

$$\mathbf{g} \;=\; +\frac{QMR^3\mu_0}{12\pi}\,\frac{\sin\theta\,\hat{\boldsymbol{\phi}}}{r^5}\,. \tag{S.17}$$

This EM momentum points in the circular $\hat{\boldsymbol{\phi}}$ direction, so the net angular momentum of the EM fields is in the $\pm z$ direction, so it's enough to calculate its z component L_z. The density if this component is

$$\ell_z \;=\; r\sin\theta\times g_\phi \;=\; \frac{QMR^3\mu_0}{12\pi}\times\frac{\sin^2\theta}{r^4}\,, \tag{S.18}$$

so all we have left to do (in this part) is to integrate this density over the volume outside the ball. Thus,

$$\begin{aligned}
L_z \;&=\; \iiint\limits_{\text{outside}} d^3\text{Vol}\,\ell_z \\
&=\; \int\limits_R^\infty dr\,r^2\times\int\limits_0^\pi d\theta\,2\pi\sin\theta\times\frac{QMR^3\mu_0}{12\pi}\,\frac{\sin^2\theta}{r^4} \\
&=\; \frac{QMR^3\mu_0}{6}\times\int\limits_R^\infty\frac{dr\,r^2}{r^4}\times\int\limits_0^\pi d\theta\,\sin^3\theta \\
&=\; \frac{QMR^3\mu_0}{6}\times\frac{1}{R}\times\frac{4}{3} \\
&=\; \tfrac{2}{9}\mu_0 QMR^2.
\end{aligned} \tag{S.19}$$

(b) Decreasing magnetization and hence time-dependent magnetic field induce electric field, which in turn exerts a torque on the electric charges at the ball's surface. And since the tangent electric field is continuous across the ball's surface, we may calculate it using either inside-the-ball or outside-the-ball formulae. But the inside-the-ball formulae are simpler, so let's use them.

Inside the ball, the magnetic field is uniform (but time-dependent)

$$\mathbf{B}(t) \;=\; \tfrac{2}{3}\mu_0 M(t)\hat{\mathbf{z}}, \tag{S.20}$$

hence induced electric field

$$\mathbf{E} \;=\; -\tfrac{1}{2}\frac{dB_z}{dt}\,r\sin\theta\,\hat{\boldsymbol{\phi}} \;=\; -\tfrac{1}{3}\mu_0\frac{dM}{dt}\,r\sin\theta\,\hat{\boldsymbol{\phi}}. \tag{S.21}$$

The torque (around z axis) this field exerts on the ball has surface area density

$$\frac{d\tau_z}{da} \;=\; R\sin\theta\times\sigma E_\phi \;=\; R\sin\theta\times\frac{Q}{4\pi R^2}\frac{-\mu_0}{3}\frac{dM}{dt}\times R\sin\theta \;=\; -\frac{Q\mu_0}{12\pi}\frac{dM}{dt}\times\sin^2\theta\,, \tag{S.22}$$

hence net torque on the ball

$$\tau_z \;=\; -\frac{Q\mu_0}{12\pi}\frac{dM}{dt}\times\iint\sin^2\theta\,d^2a \;=\; -\frac{Q\mu_0}{12\pi}\frac{dM}{dt}\times\frac{8\pi}{3}R^2 \;=\; -\tfrac{2}{9}\mu_0 QR^2\,\frac{dM}{dt}\,. \qquad \text{(S.23)}$$

The net mechanical angular momentum this torque imparts to the ball is

$$\Delta L_z \;=\; \int\tau_z\,dt \;=\; -\tfrac{2}{9}\mu_0 QR^2\int\frac{dM}{dt}\,dt \;=\; -\tfrac{2}{9}\mu_0 QR^2\bigl(M_{\rm fin}-M_{\rm init}\bigr). \qquad \text{(S.24)}$$

In particular, if we start with a non-zero magnetization $M_{\rm init}=M$ and end above the Curie point with $M_{\rm fin}=0$, then

$$\Delta L_z \;=\; +\tfrac{2}{9}\mu_0 QR^2\times M. \qquad \text{(S.25)}$$

By inspection, this is precisely the initial angular momentum (S.19) of the EM fields outside the ball, *cf.* part (a).

Problem **3**:

(a) A voltage V across the capacitor comes with capacitor charge Q, and a time-dependent charge $Q(t)$ means a current

$$I(t) \;=\; \frac{dQ}{dt} \;=\; C\times\frac{dV}{dt} \qquad \text{(S.26)}$$

through the whole circuit. In particular, this current flows through the inductor, which raises voltage

$$V_L \;=\; L\times\frac{dI}{dt} \;+\; R\times I \;=\; LC\times\frac{d^2V}{dt^2} \;+\; RC\times\frac{dV}{dt} \qquad \text{(S.27)}$$

across the inductor. By the Kirchhoff Law for voltages,

$$V_L \;+\; V_C \;=\; 0, \qquad \text{(S.28)}$$

hence

$$LC\times\frac{d^2V}{dt^2} \;+\; RC\times\frac{dV}{dt} \;+\; V \;=\; 0. \qquad \text{(S.29)}$$

Dividing this formula by LC, we get

$$\left(\frac{d^2}{dt^2} \;+\; \frac{R}{L}\times\frac{d}{dt} \;+\; \frac{1}{LC}\right)V(t) \;=\; 0, \qquad \text{(S.30)}$$

hence eq. (1) for the γ and the ω_0^2 as in eq. (2). *Quod erat demonstrandum.*

(b) The current follows from the voltage on the capacitor as

$$I(t) \;=\; C\times\frac{dV}{dt}\,, \tag{S.31}$$

where for the voltage (3)

$$\begin{aligned}
\frac{dV}{dt} \;&=\; V_0\times\bigl(-\omega'\sin(\omega' t+\phi_0)\bigr)\times e^{-\gamma t/2} \;+\; V_0\times\cos(\omega' t+\phi_0)\times\bigl(-\tfrac{1}{2}\gamma e^{-\gamma t}\bigr)\\
&=\; -\omega_0 V_0\times e^{-\gamma t/2}\times\left(\frac{\omega'}{\omega_0}\,\sin(\omega' t+\phi_0)\;+\;\frac{\gamma}{2\omega_0}\,\cos(\omega' t+\phi_0)\right)\\
&\qquad \langle\!\langle\, \text{where } \omega_0^2=\omega'^2+(\gamma/2)^2\,\rangle\!\rangle\\
&=\; -\omega_0 V_0\times e^{-\gamma t/2}\times\sin(\omega' t+\phi_0+\delta)\\
\text{for } \delta \;&=\; \arcsin\frac{\gamma}{2\omega_0}\,.
\end{aligned} \tag{S.32}$$

Consequently,

$$I(t) \;=\; -\omega_0 C V_0\times e^{-\gamma t/2}\times\sin(\omega' t+\phi_0+\delta), \tag{S.33}$$

exactly as in eq. (4).

Next, the energy stored in the LRC circuit. There is electric energy stored in the capacitor and magnetic energy stored in the inductor, and together they add up to

$$U \;=\; \tfrac{1}{2}CV^2 \;+\; \tfrac{1}{2}LI^2. \tag{S.34}$$

For the voltage (3) and the current (4), this net energy amounts to

$$\begin{aligned}
U(t) \;&=\; \tfrac{1}{2}C\times V_0^2\times e^{-\gamma t}\times\cos^2(\omega' t+\phi_0) \;+\; \tfrac{1}{2}L\times\omega_0^2C^2V_0^2\times e^{-\gamma t}\times\sin^2(\omega' t+\phi_0+\delta)\\
&\qquad \langle\!\langle\, \text{using } LC\omega_0^2=1\,\rangle\!\rangle\\
&=\; \tfrac{1}{2}CV_0^2\times e^{-\gamma t}\times\cos^2(\omega' t+\phi_0) \;+\; \tfrac{1}{2}CV_0^2\times e^{-\gamma t}\times\sin^2(\omega' t+\phi_0+\delta)\\
&=\; \tfrac{1}{2}CV_0^2\times e^{-\gamma t}\times\Bigl(\cos^2(\omega' t+\phi_0)\;+\;\sin^2(\omega' t+\phi_0+\delta)\Bigr),
\end{aligned} \tag{S.35}$$

hence in light of eq. (6) for $\alpha=\omega' t+\phi_0$,

$$U(t) \;=\; \tfrac{1}{2}CV_0^2\times e^{-\gamma t}\times\Bigl(1\;-\;\sin(\delta)\sin(2\omega' t+2\phi_0+\delta)\Bigr). \tag{S.36}$$

The overall factor $\frac{1}{2}CV_0^2$ here may be identified as U_0, the initial energy of the circuit, hence

$$U(t) \;=\; U_0\times e^{-\gamma t}\times\Big(1\,-\,\sin(\delta)\sin(2\omega' t+\text{const})\Big) \tag{S.37}$$

where $\sin\delta=(\gamma/2\omega_0)$, thus

$$U(t) \;=\; U_0\times e^{-\gamma t}\times\left(1\,-\,\frac{\gamma}{2\omega_0}\,\sin(2\omega' t+\text{const})\right), \tag{S.38}$$

exactly as on the top line of eq. (5).

Finally, for a high-quality LRC circuit with $Q\gg 1\iff\gamma\ll\omega_0$, we may neglect the small oscillations of the energy decay and approximate

$$U(t)\;\approx\;U_0\times e^{-\gamma t}\,. \tag{S.39}$$

(c) As explained in class — *cf.* http://localhost/˜vsk1958/Classes/2026s/cai.pdfmy notes, — at frequency ω, the capacitor C has imaginary impedance

$$Z_C\;=\;\frac{1}{j\omega C}\;=\;\frac{1}{-i\omega C} \tag{S.40}$$

while a perfect inductor L has imaginary impedance of opposite sign,

$$Z_L\;=\;j\omega L\;=\;-i\omega L. \tag{S.41}$$

A real inductor with some Ohmic resistance R can be thought as series sub-circuit of a perfect inductor and a resistor, thus

$$Z_{LR}\;=\;-i\omega L\;+\;R. \tag{S.42}$$

Connecting this inductor in parallel with the capacitor, we get

$$\begin{aligned}\frac{1}{Z_{LRC}}\;&=\;\frac{1}{Z_{LR}}\;+\;\frac{1}{Z_C}\;=\;\frac{1}{R-i\omega L}\;-\;i\omega C\\ &=\;\frac{1-iR\omega C-\omega^2LC}{R-i\omega L}\;=\;\frac{(1/LC)-i\omega(R/L)-\omega^2}{(1/LC)}\times\frac{1}{R-i\omega L}\\ &=\;\frac{\omega_0^2-i\gamma\omega-\omega^2}{\omega_0^2}\times\frac{1}{R-i\omega L}\end{aligned} \tag{S.43}$$

and hence LRC circuit impedance

$$Z_{LRC} = (R - \subset \omega L) \times \frac{\omega_0^2}{\omega_0^2 - \omega^2 - i\gamma\omega}. \tag{S.44}$$

exactly as in eq.(8).

(d) At frequencies ω near the resonant frequency ω_0 we may approximate

$$R - i\omega L \approx R - i\omega_0 L = R - i\sqrt{\frac{L}{C}} \tag{S.45}$$

while

$$\omega_0^2 - \omega^2 - i\gamma\omega \approx 2\omega_0(\omega_0 - \omega) - i\gamma\omega_0, \tag{S.46}$$

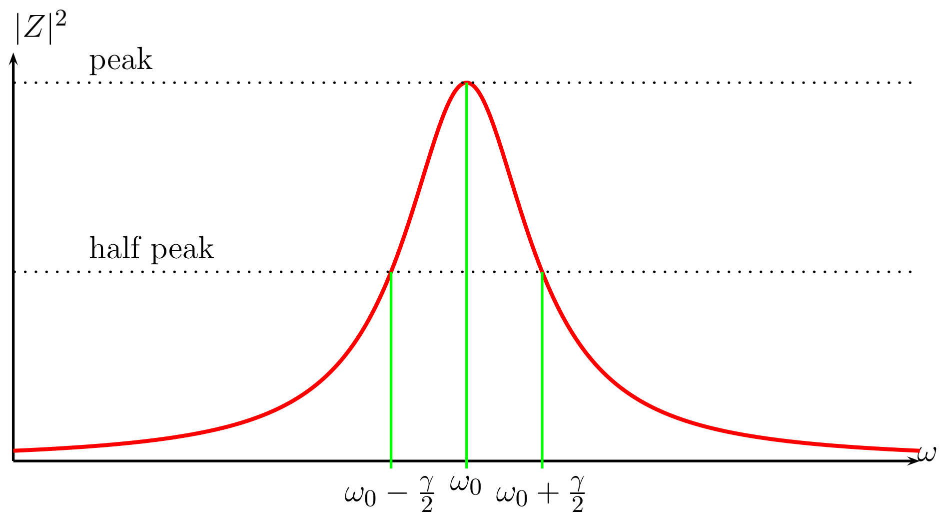
hence

$$Z \approx (R - i\sqrt{L/C}) \times \frac{\omega_0/2}{(\omega_0 - \omega) - i(\gamma/2)}, \tag{S.47}$$

and therefore

$$|Z|^2 \approx \big((L/C) + R^2\big) \times \frac{(\omega_0/2)^2}{(\omega_0 - \omega)^2 + (\gamma/2)^2}. \tag{S.48}$$

This is a Lorentzian bell curve



and it is easy to see that it peaks at ω_0 and reaches half of its peak value at

$$\omega_{\text{half peak}} = \omega_0 \pm \frac{\gamma}{2}. \tag{S.49}$$

Consequently, the width of the resonant peak is

$$\Delta\Omega = (\omega_0 + \tfrac{1}{2}\gamma) - (\omega_0 - \tfrac{1}{2}\gamma) = \gamma = \frac{\omega_0}{Q}. \tag{S.50}$$

Quod erat demonstrandum.

Problem **5**:

(a) The $f_1(z,t)$ wave has a form $g_1(z' = z - vt)$, so it's a pulse traveling right at the wave speed v. Regardless of the shape $g_1(z')$ of this pulse, it automagically obeys the wave equation because

$$\frac{\partial^2 f_1}{\partial z^2} - \frac{1}{v^2}\frac{\partial^2 f_1}{\partial t^2} = \frac{\partial^2 g_1}{\partial z'^2} - \frac{1}{v^2} \times v^2 \frac{\partial^2 g_1}{\partial z'^2} = 0. \tag{S.51}$$

Likewise $f_3(z,t) = g_3(z'' = x+vt)$ and $f_5(z,t) = g_5(z'' = z+vt)$, so both are pulses traveling left at the wave speed v, and such pulses automagically obeys the wave equation.

On the other hand, the $f_2(z,t)$ and the $f_4(z,t)$ would-be waveforms do not obey the wave equation. Indeed, for the f_2 we have

$$\begin{aligned}
f_2 &= A \times e^{-z^2/2b^2} \times \cos(\omega t), && (11)\\
\frac{\partial^2 f_2}{\partial z^2} &= A \times \frac{z^2 - b^2}{b^4}\, e^{-z^2/2b^2} \times \cos(\omega t), && (\text{S.52})\\
\frac{\partial^2 f_2}{\partial t^2} &= A \times e^{-z^2/2b^2} \times (-\omega^2)\cos(\omega t), && (\text{S.53})\\
\left(\frac{\partial^2}{\partial z^2} - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\right) f_2(z,t) &= A \times e^{-z^2/2b^2} \times \cos(\omega t) \times \left(\frac{z^2 - b^2}{b^4} + \frac{\omega^2}{v^2}\right)\\
&\neq 0, && (\text{S.54})
\end{aligned}$$

and likewise for the f_4:

$$f_4 = A \times \cos^3(kz) \times \cos(\omega t), \tag{13}$$

$$\begin{aligned}
\frac{\partial^2 f_4}{\partial z^2} &= A\times 3k^2\cos(kz)\bigl(2\sin^2(kz)-\cos^2(kz)\bigr)\times\cos(\omega t), &\text{(S.55)}\\
\frac{\partial^2 f_4}{\partial t^2} &= A\times\cos^3(kz)\times(-\omega^2)\cos(\omega t), &\text{(S.56)}\\
\left(\frac{\partial^2}{\partial z^2}-\frac{1}{v^2}\frac{\partial^2}{\partial t^2}\right)f_4(z,t) &= A\times\cos(kz)\times\cos(\omega t)\times\\
&\quad\times\left(6k^2\sin^2(kz)\,-\,3k^2\cos^2(kz)\,+\,\frac{\omega^2}{v^2}\cos^2(kz)\right)\\
&= Ak^2\times\cos(kz)\times\cos(\omega t)\times\bigl(6\sin^2(kz)\,-\,2\cos^2(kz)\bigr)\\
&\neq 0. &\text{(S.57)}
\end{aligned}$$

(b) Although the standing wave (15) is not a function of $z\pm vt$, it is a linear superposition of a wave traveling right and a wave traveling left,

$$\begin{aligned}
f_s(z,t) &= 2A\times\sin(kz)\times\cos(\omega t)\\
&= A\times\sin(kz-\omega t)\,+\,A\times\sin(kz+\omega t) &\text{(S.58)}\\
&= A\times\sin(k(z-vt))\,+\,A\times\sin(k(z+vt)).
\end{aligned}$$

Consequently, the standing wave does obey the wave equation. Indeed,

$$\begin{aligned}
\frac{\partial^2 f_s}{\partial z^2} &= A\times(-k^2)\sin(kz)\times\cos(\omega t), &\text{(S.59)}\\
\frac{\partial^2 f_s}{\partial t^2} &= A\times\sin(kz)\times(-\omega^2)\cos(\omega t), &\text{(S.60)}\\
\left(\frac{\partial^2}{\partial z^2}-\frac{1}{v^2}\frac{\partial^2}{\partial t^2}\right)f_s(z,t) &= A\times\sin(kz)\times\cos(\omega t)\times\left(-k^2\,+\,\frac{\omega^2}{v^2}\right)\\
&= 0\quad\text{for}\quad k^2\,=\,\frac{\omega^2}{v^2}\,. &\text{(S.61)}
\end{aligned}$$

Problem **6**:

(a) The wave speed in a stretched string is $v\sqrt{T/(m/\ell)}$, so the left half of the string has twice the wave speed of the right half, $v_1=2v_2$. Consequently, a general waveform $f(z,t)$

on the whole string obeys

$$\frac{\partial^2 f}{\partial z^2} - \frac{1}{v_1^2}\frac{\partial^2 f}{\partial t^2} = 0 \quad \text{for } z<0, \tag{S.62}$$

$$\text{but} \quad \frac{\partial^2 f}{\partial z^2} - \frac{1}{v_2^2}\frac{\partial^2 f}{\partial t^2} = 0 \quad \text{for } z>0, \tag{S.63}$$

$$\text{where} \quad v_1^2 = 4v_2^2. \tag{S.64}$$

Also, at $z=0$ where the two halves of the string are connected, the waveform $f(z,t)$ obeys boundary conditions:

$$@z=0,\ \text{both } f(z,t) \text{ and } \frac{\partial f}{\partial z} \text{ must be continuous functions of } z. \tag{S.65}$$

(b) On the left side of the string $z<0$, there is superposition of the incident wave running to the right and the reflected wave running back to the left,

$$f_L(z,t) = A_i \times \exp(+ik_1 z - i\omega t) + A_r \times \exp(-ik_1 z - i\omega t), \tag{S.66}$$

where to obey the wave equation for $z<0$

$$k_1 = \frac{\omega}{v_1}. \tag{S.67}$$

On the right side of the string $z>0$, there is only the transmitted wave traveling to the right, thus

$$f_R(z,t) = A_t \times \exp(+ik_2 - i\omega t), \tag{S.68}$$

but $k_2 \neq k_1$; instead,

$$k_2 = \frac{\omega}{v_2} = 2 \times k_1. \tag{S.69}$$

Now consider the boundary conditions (S.65) at $z=0$. To the immediate left at this point,

$$@z \to -0: \quad f_L(z,t) \;\to\; (A_i + A_r) \times e^{-i\omega t}, \quad \frac{\partial f_L}{\partial z} \;\to\; \bigl({}_i k_1 A_i \,-\, ik_1 A_r\bigr) \times e^{-i\omega t}, \quad \text{(S.70)}$$

while to the immediate right

$$@z \to +0: \quad f_R(z,t) \;\to\; A_t \times e^{-i\omega t}, \quad \frac{\partial f_R}{\partial z} \;\to\; +ik_2 A_t \times e^{-i\omega t}. \qquad \text{(S.71)}$$

Consequently, eqs. (S.65) become

$$\begin{aligned} A_i \;+\; A_r \;&=\; A_t\,, \\ ik_1 A_i \;-\; ik_1 A_r \;&=\; ik_2 A_t\,. \end{aligned} \qquad \text{(S.72)}$$

Solving these equations for the transmitted amplitude A_t and the reflected amplitude A_r is completely straightforward and yields

$$\begin{aligned} A_t \;&=\; \frac{2k_1}{k_1 + k_2} \times A_i \;=\; \frac{2}{3}\,A_i\,, \\ A_r \;&=\; \frac{k_1 - k_2}{k_1 + k_2} \times A_i \;=\; -\frac{1}{3}\,A_i\,. \end{aligned} \qquad \text{(S.73)}$$

(c) For a harmonic wave $f(z,t) = A\cos(kz - \omega t + \phi_0)$ on a string, the kinetic energy density is

$$u_{\text{kin}} \;=\; \frac{m/\ell}{2}\left(\frac{\partial f}{\partial t}\right)^2 \;=\; \frac{m/\ell}{2}\,A^2\omega^2 \times \sin^2(kz - \omega t + \phi_0), \qquad \text{(S.74)}$$

while the potential energy density is

$$u_{\text{pot}} \;=\; \frac{T}{2}\left(\frac{\partial f}{\partial z}\right)^2 \;=\; \frac{T}{2}\,A^2 k^2 \times \sin^2(kz - \omega t + \phi_0). \qquad \text{(S.75)}$$

Since

$$\frac{\omega^2}{k^2} \;=\; v^2 \;=\; \frac{T}{(m/\ell)}\,, \qquad \text{(S.76)}$$

the kinetic and the potential energies are equal to each other, thus the net energy density

$$u \;=\; (m/\ell) A^2\omega^2 \times \sin^2(kz - \omega t + \phi_0). \qquad \text{(S.77)}$$

And since this energy moves to the right with the wave speed v, the power transmitted by the wave is

$$P = v \times u = v(m/\ell) \times A^2\omega^2 \times \sin^2(kz - \omega t + \phi_0). \tag{S.78}$$

Time-averaging this power over the wave's period, we get

$$\langle P \rangle = v(m/ell) \times A^2\omega^2 \times \frac{1}{2} = \frac{A^2\omega^2}{2} \times \sqrt{T(m/\ell)}\,. \tag{S.79}$$

or in terms of the complex wave amplitude $A \to Ae^{i\phi_0}$,

$$\langle P \rangle = \frac{|A|^2\omega^2}{2} \times \sqrt{T(m/\ell)}. \tag{S.80}$$

In particular, for the incident wave

$$\langle P \rangle_i = \frac{|A_i|^2\omega^2}{2} \times \sqrt{T(m/\ell)_1}\,. \tag{S.81}$$

(d) Similar to eq. (S.81) for the incident wave, we have

$$\langle P \rangle_r = \frac{|A_r|^2\omega^2}{2} \times \sqrt{T(m/\ell)_1} \tag{S.82}$$

for the reflected wave, and

$$\langle P \rangle_t = \frac{|A_t|^2\omega^2}{2} \times \sqrt{T(m/\ell)_2} \tag{S.83}$$

for the transmitted wave. Comparing these powers to the incident wave power, we get

$$\frac{\langle P \rangle_r}{\langle P \rangle_i} = \frac{|A_r|^2}{|A_i|^2} = \frac{1}{9}, \tag{S.84}$$

and

$$\frac{\langle P \rangle_t}{\langle P \rangle_i} = \frac{|A_t|^2}{|A_i|^2} \times \frac{\sqrt{(m/\ell)_2}}{\sqrt{(m/ell)_1}} = \frac{4}{9} \times \frac{2}{1} = \frac{8}{9}\,. \tag{S.85}$$

Thus for the strings in question, $\frac{8}{9}$ of the incident wave power is transmitted to the right side of the connection while the remaining $\frac{1}{9}$ of the incident power is reflected back.

Problem **7**:

(a) For the amplitude (17),

$$\begin{aligned} x(z,t) &= \cos\theta \times \mathrm{Re}\big(Ce^{ikz-i\omega t}\big), \\ y(z,t) &= \sin\theta \times \mathrm{Re}\big(Ce^{ikz-i\omega t}\big), \end{aligned} \tag{S.86}$$

or in other words

$$x(z,t) = w(z,t) \times \cos\theta, \quad y(z,t) = w(z,t) \times \sin\theta \tag{S.87}$$

for the same waveform

$$w(z,t) = \mathrm{Re}\big(Ce^{ikz-i\omega t}\big) = |C| \times \cos\big(kz - \omega t + \arg(C)\big). \tag{S.88}$$

Consequently, the entire string lies in the (w,z) plane where w is a coordinate in the (x,y) plane at angle θ to the x axis, hence eqs. (S.87) for the (x,y) coordinates of the string.

(b) Now let $A = A_0 e^{-\phi_0}$ while $B = \pm iA$ as in eq. (18). Then

$$\begin{aligned} x(z,t) &= \mathrm{Re}\big(Ae^{ikz-i\omega t}\big) = A_0 \times \cos(kz - \omega t + \phi_0) \\ &= A_0 \times \cos(\omega t - kz - \phi_0) \end{aligned} \tag{S.89}$$

while

$$\begin{aligned} y(z,t) &= \mathrm{Re}\big(\pm iAe^{ikz-i\omega t}\big) = A_0 \times \cos(kz - \omega t + \phi_0 \pm 90^\circ) \\ &= \mp A_0 \times \sin(kz - \omega t + \phi_0) = \pm A_0 \times \sin(\omega t - kz - \phi_0). \end{aligned} \tag{S.90}$$

Consequently, any particular string point z moves in the (x,y) plane according to

$$\begin{aligned} x(t) &= A_0 \times \cos(\omega t - \phi), \\ y(t) &= \pm A_0 \times \sin(\omega t - \phi), \\ \text{where } \phi &= kz + \phi_0 . \end{aligned} \tag{S.91}$$

Clearly, this is a circular motion of radius A_0.

(c) In Optics convention, you look at the wave as it comes towards your eye. For a wave moving in the $+z$ direction, this means looking at the (x, y) plane from above. From this point of view, the circular motion (S.91) in the (x, y) plane is counterclockwise for the upper sign (corresponding to $B = +iA$) and clockwise for the lower sign (corresponding to $B = -iA$). Therefore,

- $B = +iA$ is the *left* circular polarization, while
- $B = -iA$ is the *right* circular polarization.

(d) Hold one end of the string in your hand and move it in a circle at a uniform rate:

$$x_{\text{end}}(t) \;=\; A_0 \times \cos(\omega t - \phi_0), \quad y_{\text{end}}(t) \;=\; \pm A_0 \times \sin(\omega t - \phi_0). \tag{S.92}$$

This will set up a circularly polarized wave on the string.