Problem **1**:

For any plane EM wave in the vacuum

$$\begin{aligned} \mathbf{E}(\mathbf{r},t) &= \mathrm{Re}\Big(\vec{\mathcal{E}}\exp(i\mathbf{k}\cdot\mathbf{r}-i\omega t)\Big), \\ \mathbf{H}(\mathbf{r},t) &= \mathrm{Re}\Big(\vec{\mathcal{H}}\exp(i\mathbf{k}\cdot\mathbf{r}-i\omega t)\Big), \\ \mathbf{k}^2 &= \frac{\omega^2}{c^2}\,. \end{aligned} \tag{S.1}$$

For the wave in question, we know the direction of the wave vector $\mathbf{k}$, hence

$$\mathbf{k} = \frac{\omega}{c\sqrt{3}}\,(1,1,1) \tag{S.2}$$

and therefore

$$\mathbf{k}\cdot\mathbf{r} = \frac{\omega}{c\sqrt{3}} * \big(x+y+z\big). \tag{S.3}$$

Next, the electric and magnetic polarization vectors $\vec{\mathcal{E}}$ and $\vec{\mathcal{H}}$ should be transverse to the wave direction $\hat{\mathbf{k}}$. Moreover, a planar (AKA linear) polarization of the wave means

$$\vec{\mathcal{E}} = \mathcal{E}_0\hat{\mathbf{e}} \quad\text{and}\quad \vec{\mathcal{H}} = \mathcal{H}_0\hat{\mathbf{h}} \tag{S.4}$$

for some *real* unit vectors $\hat{\mathbf{e}}$ and $\hat{\mathbf{h}}$ that are $\perp\hat{\mathbf{k}}$ and also $\perp$ to each other. Finally, we know the wave is polarized in the (x,y) plane, which means

$$\hat{\mathbf{e}} = (\cos\phi,\sin\phi,0) \tag{S.5}$$

for some real angle ϕ. To determine this angle, we demand that $\hat{\mathbf{e}}\perp\hat{\mathbf{k}}$, hence

$$0 = \hat{\mathbf{e}}\cdot\hat{\mathbf{k}} = \frac{1}{\sqrt{3}}\big(\cos\phi+\sin\phi+0\big) \quad\Longrightarrow\quad \phi = -45^\circ \text{ or } \phi = +135^\circ. \tag{S.6}$$

Consequently,

$$\vec{\mathcal{E}} \;=\; \frac{\mathcal{E}_0}{\sqrt{2}}\,\big(+1,-1,0\big). \tag{S.7}$$

Next, the direction of the magnetic field obtains as

$$\hat{\mathbf{h}} \;=\; \hat{\mathbf{k}}\times\hat{\mathbf{e}} \;=\; \frac{1}{\sqrt{6}}\,(1,1,1)\times(1,-1,0) \;=\; \frac{1}{\sqrt{6}}\,(1,1,-2), \tag{S.8}$$

hence

$$\vec{\mathcal{H}} \;=\; \frac{\mathcal{H}_0}{\sqrt{6}}\,(1,1,-2). \tag{S.9}$$

Also,

$$\mathcal{H}_0 \;=\; \frac{\mathcal{E}_0}{Z_0}\,, \tag{S.10}$$

where the overall complex amplitude $\mathcal{E}_0$ is undetermined by the problem data. Without loss of generality we let $\mathcal{E}_0 = E_0 e^{i\delta}$ for some real amplitude E_0, then

$$\begin{aligned}
\mathbf{E}(x,y,z,t) &= \frac{E_0}{\sqrt{2}}\exp\left(\frac{i\omega}{c\sqrt{3}}(x+y+z) \;-\; i\omega t \;+\; \delta\right) * (1,-1,0),\\
\mathbf{H}(x,y,z,t) &= \frac{E_0}{Z_0\sqrt{6}}\exp\left(\frac{i\omega}{c\sqrt{3}}(x+y+z) \;-\; i\omega t \;+\; \delta\right) * (1,1,-2).
\end{aligned} \tag{S.11}$$

Problem **2**:

The electric and the magnetic fields of the same plane wave are related to each other as

$$\mathbf{H}(\mathbf{r},t) \;=\; \frac{1}{Z_0}\,\hat{\mathbf{k}}\times\mathbf{E}(\mathbf{r},t), \quad \mathbf{E}(\mathbf{r},t) \;=\; -Z_0\,\hat{\mathbf{k}}\times\mathbf{H}(\mathbf{r},t); \tag{S.12}$$

in particular, both fields are always $\perp$ to the wave direction $\hat{\mathbf{k}}$. Now consider the stress tensor of the wave,

$$T^{ij} = \epsilon_0 E^i E^j + \mu_0 H^i H^j - \delta^{ij} u. \tag{S.13}$$

For a wave moving in the $+\hat{\mathbf{z}}$ direction,

$$H^x = \frac{+1}{Z_0} E^y, \quad H^y = \frac{-1}{Z_0} E^x \tag{S.14}$$

while $H^z = 0$ and $E^z = 0$. Consequently,

$$\begin{aligned} \mu_0 H^x H^x &= +\frac{\mu_0}{Z_0^2} E^y E^y = +\epsilon_0 E^y E^y, \\ \mu_0 H^x H^y &= -\frac{\mu_0}{Z_0^2} E^y E^x = -\epsilon_0 E^x E^y, \\ \mu_0 H^y H^y &= +\frac{\mu_0}{Z_0^2} E^x E^x = +\epsilon_0 E^x E^x, \end{aligned} \tag{S.15}$$

and therefore

$$\epsilon_0 E^i E^j + \mu_0 H^i H^j = \begin{cases} \epsilon_0 \mathbf{E}^2 & \text{for } i = j = 1 \text{ or } 2, \\ 0 & \text{for } i \neq j \text{ or } i = j = 3. \end{cases} \tag{S.16}$$

In matrix form

$$\epsilon_0 E^i E^j + \mu_0 H^i H^j = \begin{pmatrix} \epsilon_0 \mathbf{E}^2 & 0 & 0 \\ 0 & \epsilon_0 \mathbf{E}^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{S.17}$$

At the same time, the EM energy density of the wave is

$$u = \frac{\epsilon_0}{2} \mathbf{E}^2 + \frac{\mu_0}{2} \mathbf{H}^2 = \epsilon_0 \mathbf{E}^2. \tag{S.18}$$

Consequently,

$$\epsilon_0 E^i E^j + \mu_0 H^i H^j = u \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{S.19}$$

and therefore the stress tensor

$$\overleftrightarrow{T} = u\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{pmatrix} - u\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} = u\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\end{pmatrix}, \tag{S.20}$$

or in indices

$$T^{ij} = -u\hat{k}^i\hat{k}^j. \tag{S.21}$$

Note: this is the instantaneous stress tensor of the wave in terms of its instantaneous energy density at the same place and time. The matrix structure of this stress tensor depends on the direction $\hat{\mathbf{k}}$ of the EM wave, but it does not depend on the directions of the electric or magnetic fields, only on their magnitudes (via the energy density u).

The time-averaging the stress tensor (S.21) is completely straightforward: its matrix structure is constant, so all we need to do is to time-average the energy density factor, thus

$$\left\langle\overleftrightarrow{T}\right\rangle = -\langle u\rangle * \hat{k}^i\hat{k}^j. \tag{S.22}$$

Or in terms of the wave's intensity

$$I = c\langle u\rangle = \frac{|\vec{\mathcal{E}}|^2}{2Z_0},$$

the stress tensor is

$$\left\langle T^{ij}\right\rangle = -\frac{I}{c} * \hat{k}^i\hat{k}^j \xrightarrow[\text{for }\hat{\mathbf{k}}=\hat{\mathbf{z}}]{} \frac{I}{c}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\end{pmatrix}. \tag{S.23}$$

Problem **3**:

(a–b) Inside the reflector at $z > 0$, there is no electric or magnetic fields, $\mathbf{E} = 0$ and $\mathbf{H} = 0$. OOH, in the free space left of the boundary (*i.e.*, at $z < 0$), the electric field (2) is accompanied by the magnetic field

$$\mathbf{H}(z,t) = \frac{\hat{\mathbf{k}} \times \vec{\mathcal{E}}_i}{Z_0} \exp(+ikz - i\omega t) - \frac{\hat{\mathbf{k}} \times \vec{\mathcal{E}}_r}{Z_0} \exp(-ikz - i\omega t). \tag{S.24}$$

Note the red minus sign here to opposite directions of the incident and reflected waves.

Next, the boundary conditions. The electric field (2) is tangent to the boundary, so it must be continuous across $z = 0$. This requires

$$\mathbf{E}[\text{eq. (2)}] \to 0 \quad \text{for } z \to -0, \tag{S.25}$$

hence

$$\big(\mathbf{E}_i + \mathbf{E}_r\big) \exp(-i\omega t) = 0, \tag{S.26}$$

and therefore $\vec{\mathcal{E}}_r = -\vec{\mathcal{E}}_i$.

On the other hand, the magnetic field $\mathbf{H}$, — or rather its tangent components H_x and H_y — may be discontinuous if there is a surface current (4) along the boundary:

$$\hat{\mathbf{z}} \times \operatorname{disc}(\mathbf{H}) = \mathbf{K}, \tag{S.27}$$

or

$$-\operatorname{disc}(\mathbf{H}) \stackrel{\text{def}}{=} \mathbf{H}(z=-0) - \mathbf{H}(z=+0) = \hat{\mathbf{z}} \times \mathbf{K}. \tag{S.28}$$

In terms of the magnetic field (S.24) for $z < 0$ and $\mathbf{H} = 0$ for $z > 0$, this means

$$\frac{1}{Z_0} (\hat{\mathbf{k}} = \hat{\mathbf{z}}) \times \big(\vec{\mathcal{E}}_i - \vec{\mathcal{E}}_r\big) e^{-i\omega t} = \hat{\mathbf{z}} \times \vec{\mathcal{K}} e^{-i\omega t} \tag{S.29}$$

and consequently the surface current amplitude

$$\vec{\mathcal{K}} = \frac{\vec{\mathcal{E}}_i - \vec{\mathcal{E}}_r}{Z_0}. \tag{S.30}$$

Moreover, in light of the electric boundary condition $\vec{\mathcal{E}}_r = -\vec{\mathcal{E}}_i$,

$$\vec{\mathcal{K}} = \frac{2}{Z_0}\,\vec{\mathcal{E}}_i\,. \tag{S.31}$$

(c) A current in a magnetic field feels the Ampere force. In particular, a surface current density feels *force density*

$$\frac{\mathbf{F}}{\text{area}} = \mathbf{K}\times\mathbf{B}. \tag{S.32}$$

However, once should keep two caveats in mind when calculating this force. First, the magnetic field is discontinuous across the current sheet, so $\mathbf{B}$ in eq. (S.32) is the average of the magnetic fields at the two sides,

$$\frac{\mathbf{F}}{\text{area}} = \mathbf{K}\times\frac{\mathbf{B}_{\text{left}}+\mathbf{B}_{\text{right}}}{2}, \tag{S.33}$$

which for the problem at hand means

$$\frac{\mathbf{F}}{\text{area}} = \mathbf{K}\times\frac{\mu_0}{2}\mathbf{H}(z=-0). \tag{S.34}$$

Second, eq. (S.34) applies to the instantaneous forces in terms of instantaneous currents and magnetic fields. But like any bilinear formula, it can be generalized to give the time-averaged force in terms of the amplitudes of harmonic currents and fields, thus

$$\frac{\langle\mathbf{F}\rangle}{\text{area}} = \frac{\mu_0}{2} * \frac{1}{2}\operatorname{Re}\bigl(\vec{\mathcal{K}}^* \times \vec{\mathcal{H}}(z=-0)\bigr). \tag{S.35}$$

As we saw above,

$$\vec{\mathcal{H}}(z=-0) = \frac{1}{Z_0}\hat{\mathbf{z}}\times(\vec{\mathcal{E}}_i - \vec{\mathcal{E}}_r) = \frac{2}{Z_0}\,\hat{\mathbf{z}}\times\vec{\mathcal{E}}_i \tag{S.36}$$

while

$$\vec{\mathcal{K}} = \frac{2}{Z_0}\,\vec{\mathcal{E}}_i\,. \tag{S.37}$$

Consequently,

$$\vec{\mathcal{K}}^* \times \vec{\mathcal{H}}(z=-0) \;=\; \frac{4}{Z_0^2}\,\vec{\mathcal{E}}_i^* \times (\hat{\mathbf{z}} \times \vec{\mathcal{E}}_i) \;=\; \frac{4}{Z_0^2}\Big(\hat{\mathbf{z}}\big(\vec{\mathcal{E}}_i^* \cdot \vec{\mathcal{E}}_i\big) \,-\, \vec{\mathcal{E}}_i\big(\vec{\mathcal{E}}_i^* \cdot \hat{\mathbf{z}}\big)\Big) \;=\; \frac{4}{Z_0^2}\Big(\hat{\mathbf{z}}\left|\vec{\mathcal{E}}_i\right|^2 \,-\, \mathbf{0}\Big) \tag{S.38}$$

because $\big(\vec{\mathcal{E}}_i^* \cdot \hat{\mathbf{z}}\big) = (\vec{\mathcal{E}}_i \cdot \hat{\mathbf{z}}) = 0$ (the $\vec{\mathcal{E}}_i$ amplitude is transverse), hence

$$\mathrm{Re}\big(\vec{\mathcal{K}}^* \times \vec{\mathcal{H}}(z=-0)\big) \;=\; \frac{4}{Z_0^2}\left|\vec{\mathcal{E}}_i\right|^2 \hat{\mathbf{z}}, \tag{S.39}$$

and therefore

$$\frac{\langle \mathbf{F}\rangle}{\text{area}} \;=\; \frac{\mu_0}{4}\,\mathrm{Re}\big(\vec{\mathcal{K}}^* \times \vec{\mathcal{H}}(z=-0)\big) \;=\; \frac{\mu_0}{Z_0^2}\left|\vec{\mathcal{E}}_i\right|^2 \hat{\mathbf{z}} \;=\; \epsilon_0 \left|\vec{\mathcal{E}}_i\right|^2 \hat{\mathbf{z}}\,. \tag{S.40}$$

Or in terms of the incident wave's intensity

$$I \;=\; c u_{\text{incident}} \;=\; \frac{c\epsilon_0}{2}\left|\vec{\mathcal{E}}_i\right|^2, \tag{S.41}$$

the force on the surface current — and hence on the reflector — is

$$\frac{\langle \mathbf{F}\rangle}{\text{area}} \;=\; \frac{2I}{c}\,\hat{\mathbf{z}}, \tag{S.42}$$

which means

$$\text{radiation pressure} \;=\; \frac{2I}{c}\,. \tag{S.43}$$

<u>Problem **4**</u>:

(a) The first filter — regardless of its specific polazization axis — transmits one half of the unpolarized light's intensity, $I_1 = \frac{1}{2}I_0$. After that, the fraction of light's intensity going through each subsequent filter follows from the Malus Law: Each filter's axis is tilted 45° relative to the previous filter's axis and hence from the polarization axis of the light emerging from that previous filter, so the fraction going through the current filter is $\cos^2 45° = \frac{1}{2}$. Thus, $I_2 = \frac{1}{2}I_1$, $I_3 = \frac{1}{2}I_2$, $I_4 = \frac{1}{2}I_3$, and therefore

$$I_1 \;=\; \tfrac{1}{2}I_0\,, \quad I_2 \;=\; \tfrac{1}{4}I_0\,, \quad I_3 \;=\; \tfrac{1}{8}I_0\,, \quad I_4 \;=\; \tfrac{1}{16}I_0\,. \tag{S.44}$$

(b) For the second sequence, the first filter is circularly rather than linearly polarized, but just the same it transmits 50% of the inpolarized light's intensity, $I_1 = \frac{1}{2}I_0$. But after that, for each subsequent filter we need the generalized Malus Law:

$$\frac{I_n}{I_{n-1}} = \left|\mathbf{e}_n^* \cdot \mathbf{e}_{n-1}\right|^2 \tag{S.45}$$

where $\mathbf{e}_n$ is the unit polarization vector for the filter#n. For the sequence at hand,

$$\begin{aligned} \mathbf{e}_1 &= \frac{1}{\sqrt{2}}(1,+i), \\ \mathbf{e}_2 &= (0,1), \\ \mathbf{e}_3 &= \frac{1}{\sqrt{2}}(1,+i), \\ \mathbf{e}_4 &= (1,0), \end{aligned} \tag{S.46}$$

hence

$$\begin{aligned} \left|\mathbf{e}_2^* \cdot \mathbf{e}_1\right|^2 &= \tfrac{1}{2}, \\ \left|\mathbf{e}_3^* \cdot \mathbf{e}_2\right|^2 &= \tfrac{1}{2}, \\ \left|\mathbf{e}_4^* \cdot \mathbf{e}_3\right|^2 &= \tfrac{1}{2}. \end{aligned} \tag{S.47}$$

Therefore, similar to the first sequence $I_2 = \frac{1}{2}I_1$, $I_3 = \frac{1}{2}I_2$, $I_4 = \frac{1}{2}I_3$, and hence

$$I_1 = \tfrac{1}{2}I_0\,, \quad I_2 = \tfrac{1}{4}I_0\,, \quad I_3 = \tfrac{1}{8}I_0\,, \quad I_4 = \tfrac{1}{16}I_0\,. \tag{S.48}$$

Problem **5**:

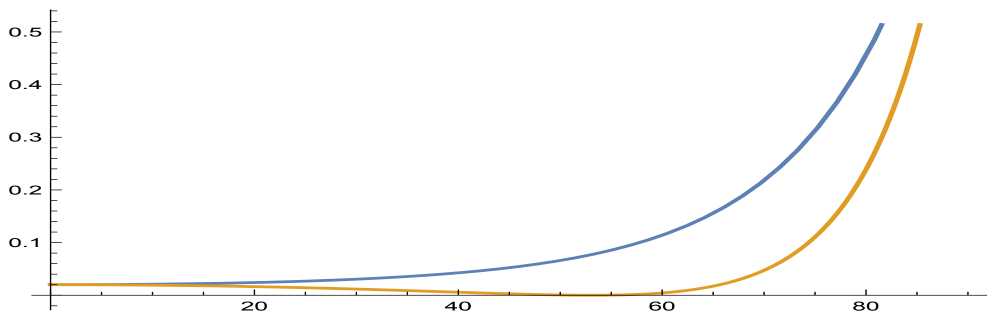
My notes on refraction and reflection of EM waves gives explicit formulae for the reflectivity of a boundary between two transparent materials as a function of the incidence angle α, see eqs. (86) and (88) on page 18 of the notes. The reflectivity depends on the EM wave's polarization: For a wave polarized normally to the plane of incidence

$$R_{\perp} = \frac{\left(\sqrt{(n_2/n_1)^2 - \sin^2\alpha} - \cos\alpha\right)^2}{\left(\sqrt{(n_2/n_1)^2 - \sin^2\alpha} + \cos\alpha\right)^2}, \tag{S.49}$$

while for a wave polarized within the plane of incidence

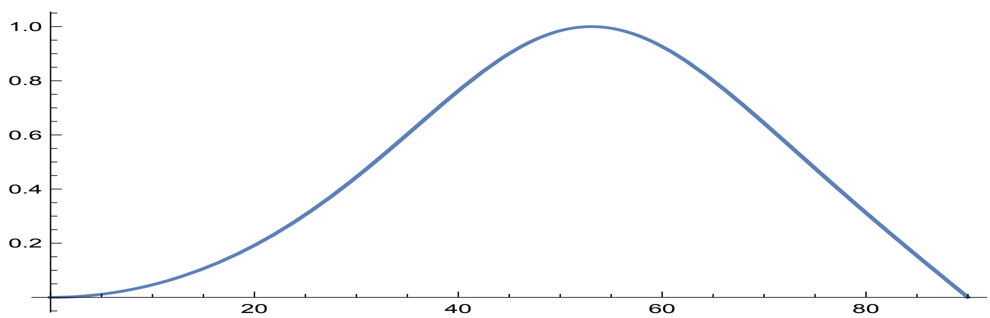
$$R_{\|} = \frac{\left(\sqrt{(n_2/n_1)^2 - \sin^2\alpha} - (n_2/n_1)^2\cos\alpha\right)^2}{\left(\sqrt{(n_2/n_1)^2 - \sin^2\alpha} + (n_2/n_1)^2\cos\alpha\right)^2}. \tag{S.50}$$

To plot these reflectivities as function of the incidence angle, I used Mathematica: Here is my code, and here are the plots:



(The blue line is the $R_{\perp}(\alpha)$ for the polarization normal to the plane of incidence, and the orange line is the $R_{\|}(\alpha)$ for the polarization within the plane of incidence.)

Finally, here is the plot of the degree of polarization (5):



Note 100% polarization of the reflected light at the Brewster angle

$$\alpha_B \;=\; \arctan\frac{n_2}{n_1} \;\approx\; 53^\circ. \tag{S.51}$$

Problem **6**:

(a) For each of the 5 plane waves (7), the electric field

$$\mathbf{E}_j(\mathbf{r},t) \;=\; \vec{\mathcal{E}}_j \exp(i\mathbf{k}_j\cdot\mathbf{r} - i\omega t) \tag{S.52}$$

comes with the magnetic field

$$\begin{aligned} \mathbf{H}_j(\mathbf{r},t) \;&=\; \vec{\mathcal{H}}_j \exp(i\mathbf{k}_j\cdot\mathbf{r} - i\omega t) \\ \text{for } \vec{\mathcal{H}}_j \;&=\; \frac{\hat{\mathbf{k}}_j}{Z_j}\times\vec{\mathcal{E}}_j \;=\; \frac{n_j\hat{\mathbf{k}}_j}{Z_0}\times\vec{\mathcal{E}}_j\,. \end{aligned} \tag{S.53}$$

Specifically:

$$\begin{aligned} \text{for } z<0: \quad Z_0\mathbf{H}(z,t) \;&=\; n_w\hat{\mathbf{z}}\times\vec{\mathcal{E}}_1\,\exp(+ik_wz-i\omega t) \;-\; n_w\hat{\mathbf{z}}\times\vec{\mathcal{E}}_2\,\exp(-ik_wz-i\omega t), \\ \text{for } 0<z<D: \quad Z_0\mathbf{H}(z,t) \;&=\; n_o\hat{\mathbf{z}}\times\vec{\mathcal{E}}_3\,\exp(+ik_oz-i\omega t) \;-\; n_o\hat{\mathbf{z}}\times\vec{\mathcal{E}}_4\,\exp(-ik_oz-i\omega t), \\ \text{but for } z>D: \quad Z_0\mathbf{H}(z,t) \;&=\; n_a\hat{\mathbf{z}}\times\vec{\mathcal{E}}_5\,\exp(+ik_az-i\omega t) \;+\; 0. \end{aligned} \tag{S.54}$$

Next, the boundary conditions. All the electric and magnetic fields (7) and (S.54) are parallel to the boundaries, and there are no conduction currents on any of the boundaries. Consequently, both $\mathbf{E}(z,t)$ and $\mathbf{H}(z,t)$ fields must be continuous at both $z=0$ and $z=D$ boundaries. For the electric fields (7), this means

$$\begin{aligned} \vec{\mathcal{E}}_1 \;+\; \vec{\mathcal{E}}_2 \;&=\; \vec{\mathcal{E}}_3 \;+\; \vec{\mathcal{E}}_4\,, \\ \vec{\mathcal{E}}_3\exp(+ik_oD) \;+\; \vec{\mathcal{E}}_4\exp(-ik_oD) \;&=\; \vec{\mathcal{E}}_5\exp(+ik_aD), \end{aligned} \tag{S.55}$$

while for the magnetic fields (S.54) we have

$$\begin{aligned} n_w\hat{\mathbf{z}}\times\vec{\mathcal{E}}_1 \;-\; n_w\hat{\mathbf{z}}\times\vec{\mathcal{E}}_2 \;&=\; n_o\hat{\mathbf{z}}\times\vec{\mathcal{E}}_3 \;-\; n_o\hat{\mathbf{z}}\times\vec{\mathcal{E}}_4\,, \\ n_o\hat{\mathbf{z}}\times\vec{\mathcal{E}}_3\exp(+ik_oD) \;-\; n_o\hat{\mathbf{z}}\times\vec{\mathcal{E}}_4\exp(-ik_oD) \;&=\; n_a\hat{\mathbf{z}}\times\vec{\mathcal{E}}_5\exp(+ik_aD), \end{aligned} \tag{S.56}$$

and hence

$$\begin{aligned} \vec{\mathcal{E}}_1 \;-\; \vec{\mathcal{E}}_2 \;&=\; \frac{n_o}{n_w}\Big(\vec{\mathcal{E}}_3 \,-\, \vec{\mathcal{E}}_4\big), \\ \vec{\mathcal{E}}_3\exp(+ik_oD) \;-\; \vec{\mathcal{E}}_4\exp(-ik_oD) \;&=\; \frac{n_a}{n_o}\vec{\mathcal{E}}_5\exp(+ik_aD). \end{aligned} \tag{S.57}$$

(b–c) Solving the system of 4 linear equations (S.55) and (S.57) is best done on a computer, using Mathematica or equivalent formulae. Here is my Mathematica code. To simplify notations, I use $\phi = k_0 \times D$ and $\delta = k_a \times D$.

As you can see from my code — or rather from the Mathematica's answers to it, — the reflectivity and the transmissivity indeed ads up to 1. Also, the reflectivity and the transmissivity oscillate as functions of the oil slick's optical thickness k_oD. Specifically,

$$\frac{1}{T} \;=\; \frac{(n_a+n_w)^2}{4n_an_w}\times\cos^2(k_oD) \;+\; \frac{(n_o^2+n_an_w)^2}{4n_o^2n_an_w}\times\sin^2(k_oD). \tag{S.58}$$

for $n_a \approx 1$, $n_w \approx 1.33$, and $n_o \approx 1.5$, this formula evaluates to

$$\frac{1}{T} \;\approx\; 1.02\times\cos^2(k_oD) \;+\; 1.07\times\sin^2(k_oD), \tag{S.59}$$

hence

$$R \;=\; 1 \;-\; T \;\approx\; 0.02\cos^2(k_oD) \;+\; 0.07\sin^2(k_oD) \tag{S.60}$$

which varies between about 2% and 7%, depending on the oil slick thickness: the reflectivity is minimal for integer k_oD/π and maximal for half-integer k_oD/π. For the yellow light of vacuum wavelength 555 nm, the wavelength in oil is

$$\lambda_0 \;=\; \frac{2\pi}{k_o} \;=\; \frac{\lambda_{\rm vac}}{n_o} \;=\; 370\text{ nm}, \tag{S.61}$$

hence minimal reflectivity about 2% for D being integer multiple of 185 nm, and maximal reflectivity about 7% for D being half-integer multiple of 185 nm.

Problem **7**:

(a) For a harmonic electric field

$$\mathbf{E}(\mathbf{r},t) \;=\; \mathbf{F}(\mathbf{r})\, e^{-i\omega t} \tag{S.62}$$

like in eq. (9), the wave equation

$$\left(\nabla^2 \;-\; \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{E}(\mathbf{r},t) \;=\; 0 \tag{S.63}$$

becomes

$$\nabla^2\mathbf{F}(\mathbf{r}) * e^{-i\omega t} \;+\; \mathbf{F}(\mathbf{r}) * \frac{\omega^2}{c^2}\, e^{-i\omega t} \;=\; 0 \tag{S.64}$$

and hence

$$\left(\nabla^2 \,+\, k^2\right)\mathbf{F}(\mathbf{r}) \;=\; 0. \tag{S.65}$$

Moreover, for the electric field in eq. (9),

$$\mathbf{F}(r,\theta,\phi) \;=\; f(r) * \sin(\theta)\hat{\boldsymbol{\phi}} \tag{S.66}$$

as in eq. (10) for

$$f(r) \;=\; \frac{A}{r}\left(1 \,+\, \frac{i}{kr}\right) e^{ikr}, \tag{S.67}$$

hence in light of eq. (10)

$$\left(\nabla^2 \,+\, k^2\right)\mathbf{F}(r,\theta,\phi) \;=\; \left(f''(r) \,+\, \frac{2}{r}\, f'(r) \,-\, \frac{2}{r^2}\, f(r) \,+\, k^2 f(r)\right) * \sin(\theta)\hat{\boldsymbol{\phi}}, \tag{S.68}$$

so the wave equation reduced to the ordinary differential equation for the radial profile $f(r)$, namely

$$f''(r) \,+\, \frac{2}{r}\, f'(r) \,-\, \frac{2}{r^2}\, f(r) \,+\, k^2 f(r) \;=\; 0. \tag{S.69}$$

Now let's check this equation for the $f(r)$ as in eq. (S.67):

$$f(r) \;=\; A\left(\frac{1}{r} \,+\, \frac{i}{kr^2}\right) e^{ikr}, \tag{S.67}$$

$$\begin{aligned} f'(r) &= A\left(\frac{ik}{r} - \frac{2}{r^2} - \frac{2i}{kr^3}\right)e^{ikr}, && \text{(S.70)}\\ f''(r) &= A\left(-\frac{k^2}{r} - \frac{3ik}{r^2} + \frac{6}{r^3} + \frac{6i}{kr^4}\right)e^{ikr}, && \text{(S.71)} \end{aligned}$$

hence

$$\begin{aligned} f''(r) + \frac{2}{r}f'(r) - \frac{2}{r^2}f(r) + k^2 f(r) &= Ae^{ikr}(\cdots) \\ \text{where } (\cdots) &= -\frac{k^2}{r} - \frac{3ik}{r^2} + \frac{6}{r^3} + \frac{6i}{kr^4} \\ &\quad + \frac{2ik}{r^2} - \frac{4}{r^3} - \frac{4i}{kr^4} \\ &\quad - \frac{2}{r^3} - \frac{2i}{kr^4} + \frac{k^2}{r} + \frac{ik}{r^2} \\ &= 0. \end{aligned} \tag{S.72}$$

Thus, the electric field (9) indeed obeys the wave equation.

(b) Inside the textbook front cover there are equations for gradients, divergences, curls, and Laplacians of fields in Cartesian, spherical, and cylindrical coordinates. In particular, in spherical coordinates

$$\nabla\cdot\mathbf{F} = \frac{1}{r^2}\frac{\partial}{\partial r}\big(r^2 F_r\big) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\big(\sin\theta\,F_\theta\big) + \frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial\phi}, \tag{S.73}$$

$$\begin{aligned} \nabla\times\mathbf{F} &= \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\big(\sin\theta\,F_\phi\big) - \frac{\partial F_\theta}{\partial\phi}\right]\mathbf{n} \\ &\quad + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial F_r}{\partial\phi} - \frac{\partial}{\partial r}\big(rF_\phi\big)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}\big(rF_\theta\big) - \frac{\partial F_r}{\partial\theta}\right]\hat{\boldsymbol{\phi}}. \end{aligned} \tag{S.74}$$

The electric field (9) always points in the $\hat{\boldsymbol{\phi}}$ direction, but the value of the E_ϕ component does not depend on the ϕ, hence according to eq. (S.73) the divergence $\nabla\cdot\mathbf{E}$ automagically vanishes. This verifies the Gauss Law $\nabla\cdot\mathbf{E} = 0$ in the absence of any charges.

Next, the curl $\nabla\times\mathbf{E}$. For vector fields of the form (10),

$$\nabla\times\mathbf{F} = \frac{2f(r)}{r}\cos\theta\,\mathbf{n} - \left(f'(r) + \frac{f(r)}{r}\right)\sin\theta\,\hat{\boldsymbol{\theta}}. \tag{S.75}$$

In the context of the electric field (9), we have

$$\frac{f(r)}{r} \;=\; A\left(\frac{1}{r^2}\,+\,\frac{i}{kr^3}\right)e^{ikr-i\omega t}, \tag{S.76}$$

$$f'(r)\,+\,\frac{f(r)}{r} \;=\; A\left(\frac{ik}{r}\,-\,\frac{1}{r^2}\,-\,\frac{i}{kr^3}\right)e^{ikr-i\omega t}, \tag{S.77}$$

hence

$$\begin{aligned}\nabla\times\mathbf{E} \;&=\; Ae^{ikr-i\omega t}\left[2\left(\frac{1}{r^2}\,+\,\frac{i}{kr^3}\right)\cos\theta\,\mathbf{n}\,-\,\left(\frac{ik}{r}\,-\,\frac{1}{r^2}\,-\,\frac{i}{kr^3}\right)\sin\theta\,\hat{\boldsymbol{\theta}}\right]\\ &=\; \frac{-ikA}{r}\,e^{ikr-i\omega t}\left[\sin\theta\,\hat{\boldsymbol{\theta}}\,+\,\frac{i}{rk}\left(1+\frac{i}{kr}\right)\left(\sin\theta\,\hat{\boldsymbol{\theta}}\,+\,2\cos\theta\,\mathbf{n}\right)\right].\end{aligned} \tag{S.78}$$

At the same time, for a harmonic magnetic field of frequency ω,

$$-\frac{\partial\mathbf{B}}{\partial t} \;=\; +i\omega\mathbf{B} \tag{S.79}$$

hence for the magnetic field (11)

$$-\frac{\partial\mathbf{B}}{\partial t} \;=\; \frac{-i\omega A}{rc}\,e^{ikr-i\omega t}\left[\sin\theta\,\hat{\boldsymbol{\theta}}\,+\,\frac{i}{rk}\left(1+\frac{i}{kr}\right)\left(\sin\theta\,\hat{\boldsymbol{\theta}}\,+\,2\cos\theta\,\mathbf{n}\right)\right]. \tag{S.80}$$

Comparing this formula to eq. (S.78) and identifying $(\omega/c) = k$, we immediately see that the electric and the magnetic field obey the Induction Law

$$\nabla\times\mathbf{E} \;=\; -\frac{\partial\mathbf{B}}{\partial t}\,. \tag{S.81}$$

The Magnetic Gauss Law $\nabla\cdot\mathbf{B} = 0$ follows immediately from this formula: On one hand,

$$\nabla\cdot\frac{\partial\mathbf{B}}{\partial t} \;=\; -\nabla\cdot(\nabla\times\mathbf{E}) \;=\; 0, \tag{S.82}$$

on the other hand,

$$\nabla\cdot\frac{\partial\mathbf{B}}{\partial t} \;=\; \nabla\cdot\big(-i\omega\mathbf{B}\big) \;=\; -i\omega\nabla\cdot\mathbf{B}, \tag{S.83}$$

hence $\subset \omega\nabla\cdot\mathbf{B} = 0$ and therefore $\nabla\cdot\mathbf{B} = 0$.

Finally, the Maxwell–Ampere Law

$$\nabla\times\mathbf{B} \;=\; \frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t}\,. \tag{S.84}$$

On the RHS of this formula, we have

$$\frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t} \;=\; \frac{-i\omega}{c^2}\,\mathbf{E} \tag{S.85}$$

and hence

$$\frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t} \;=\; \frac{-i\omega A}{c^2 r}\,e^{ikr-i\omega t}\left(1\,+\,\frac{i}{kr}\right)*\sin\theta\,\hat{\boldsymbol{\phi}}. \tag{S.86}$$

Now consider the magnetic field (11) and hence $\nabla\times\mathbf{B}$ on the LHS of the Maxwell–Ampere Law. In spherical components,

$$\begin{aligned}
B_r &= -\frac{A}{cr}\,e^{ikr-i\omega t}\left(\frac{2i}{kr}\,-\,\frac{2}{k^2r^2}\right)\cos\theta,\\
B_\theta &= -\frac{A}{cr}\,e^{ikr-i\omega t}\left(1\,+\,\frac{i}{kr}\,-\,\frac{1}{k^2r^2}\right)\sin\theta,\\
B_\phi &= 0;
\end{aligned} \tag{S.87}$$

in particular, $B_\phi=0$ while B_r and B_θ do not depend on ϕ. Consequently, according to the curl eqs. (S.74), the curl components $(\nabla\times\mathbf{B})_r$ and $(\nabla\times\mathbf{B})_\theta$ automagically vanish, which leaves us with

$$\nabla\times\mathbf{B} \;=\; (\nabla\times\mathbf{B})_\phi\hat{\boldsymbol{\phi}} \tag{S.88}$$

for

$$(\nabla\times\mathbf{B})_\phi \;=\; \frac{1}{r}\left[\frac{\partial(rB_\theta)}{\partial r}\,-\,\frac{\partial B_r}{\partial\theta}\right]. \tag{S.89}$$

For the components (S.87),

$$\frac{1}{r}\frac{\partial(rB_\theta)}{\partial r} \;=\; -\frac{A}{rc}\,e^{ikr-i\omega c}\left[ik\,-\,\frac{1}{r}\,-\,\frac{2i}{kr^2}\,+\,\frac{2}{k^2r^3}\right]\sin\theta, \tag{S.90}$$

$$-\frac{1}{r}\frac{\partial B_r}{\partial\theta} \;=\; -\frac{A}{rc}\,e^{ikr-i\omega c}\left[\frac{2i}{kr^2}\,-\,\frac{2}{k^2r^3}\right]\sin\theta, \tag{S.91}$$

hence altogether

$$\begin{aligned}\nabla\times\mathbf{B} &= -\frac{A}{rc}\,e^{ikr-i\omega c}\left[ik\,-\,\frac{1}{r}\right]*\sin\theta\,\hat{\boldsymbol{\phi}}\\ &= -\frac{ikA}{rc}\,e^{ikr-i\omega c}\left(1\,+\,\frac{i}{kr}\right)*\sin\theta\,\hat{\boldsymbol{\phi}}.\end{aligned}\tag{S.92}$$

Comparing this formula to eq. (S.86) and using $(k/c)=(\omega/c^2)$, we immediately see that

$$\nabla\times\mathbf{B} \;=\; \frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t}\,,\tag{S.93}$$

which verifies the Maxwell Ampere equation.

Altogether, we see that the electric field (9) and the magnetic field (11) obey all 4 of the Maxwell equations. *Quod erat demonstrandum.*

(c) Eqs. (9) and (11) for the electric and the magnetic fields have general form

$$\mathbf{E}(\mathbf{r},t) \;=\; \mathrm{Re}\Big(\mathbf{E}_0(\mathbf{r})\,e^{-i\omega t}\Big),\qquad \mathbf{B}(\mathbf{r},t) \;=\; \mathrm{Re}\Big(\mathbf{B}_0(\mathbf{r})\,e^{-i\omega t}\Big),\tag{S.94}$$

so the time-averaged Poynting vector obtains as

$$\langle\mathbf{S}\rangle \;=\; \frac{1}{\mu_0}\,\langle\mathbf{E}\times\mathbf{B}\rangle \;=\; \frac{1}{2\mu_0}\,\mathrm{Re}\Big(\mathbf{E}_0^*\times\mathbf{B}_0\Big).\tag{S.95}$$

Specifically,

$$\mathbf{E}_0^*(r,\theta,\phi) \;=\; \frac{A^*}{r}\,e^{-ikr}\left(1-\frac{i}{kr}\right)*\sin\theta\,\hat{\boldsymbol{\phi}},\tag{S.96}$$

$$\mathbf{B}_0(r,\theta,\phi) \;=\; -\frac{A}{rc}\,e^{+ikr}\left[\left(1+\frac{i}{kr}-\frac{1}{k^2r^2}\right)*\sin\theta\,\hat{\boldsymbol{\theta}}\;+\;\left(\frac{2i}{kr}-\frac{2}{k^2r^2}\right)*\cos\theta\,\mathbf{n}\right],\tag{S.97}$$

hence

$$\mathbf{E}_0^*\times\mathbf{B}_0 \;=\; -\frac{|A|^2}{cr^2}\left[\begin{array}{c}\left(1+\dfrac{i}{(kr)^3}\right)*\sin^2\theta*(\hat{\boldsymbol{\phi}}\times\hat{\boldsymbol{\theta}}\;=\;-\mathbf{n})\\ +\;\dfrac{2i}{kr}\left(1+\dfrac{1}{(kr)^2}\right)*\sin\theta\cos\theta*(\hat{\boldsymbol{\phi}}\times\mathbf{n}\;=\;\hat{\boldsymbol{\theta}})\end{array}\right].\tag{S.98}$$

Taking the real part of this expression, we get

$$\mathrm{Re}\Big(\mathbf{E}_0^* \times \mathbf{B}_0\Big) \;=\; +\frac{|A|^2}{cr^2}\Big[1 * \sin^2\theta * \mathbf{n} \,+\, 0 * \sin(2\theta) * \hat{\boldsymbol{\theta}} \;=\; \sin^2\theta * \mathbf{n}\Big], \tag{S.99}$$

hence

$$\langle \mathbf{S} \rangle \;=\; \frac{|A|^2}{2\mu_0 c}\,\frac{\sin^2\theta}{r^2}\,\mathbf{n} \;=\; \frac{|A|^2}{2Z_0}\,\frac{\sin^2\theta}{r^2}\,\mathbf{n}. \tag{S.100}$$

By inspection, this time-averaged Pointing vector points in the radial direction $+\mathbf{n}$, while it's magnitude scales with the radius as $1/r^2$.

(d) Consider a spherical surface centered at the origin of the wave in question. Take an element of this surface contained in a cone of solid angle $d\Omega$, then the vector area of this element is

$$\mathbf{d^2a} \;=\; R^2(d\Omega)\mathbf{n}, \tag{S.101}$$

so the wave power flowing through this area is

$$\langle \mathbf{S} \rangle \cdot \mathbf{d^2a} \;=\; \frac{|A|^2}{2Z_0}\,\frac{\sin^2\theta}{R^2}\,\mathbf{n} \cdot R^2(d\Omega)\mathbf{n} \;=\; \frac{|A|^2}{2Z_0}\,\sin^2\theta\, d\Omega. \tag{S.102}$$

Note: this power does not depend on the radius R. Physically, it means that the power is emitted into a cone of solid angle d^Ω and then spreads out radially, so the same net power remains in the same cone $d\Omega$. However, different powers are emitted in different directions, specifically power per solid angle

$$\frac{dP}{d\Omega} \;=\; \frac{|A|^2}{2Z_0}\,\sin^2\theta. \tag{S.103}$$

To get the net power of the spherical wave, we should simply integrate this power per solid angle over 4π directions. In spherical coordinates $d^2\Omega = \sin\theta\, d\theta\, d\phi$, hence

$$\oiint\limits_{4\pi} \sin^2\theta\, d^2\Omega \;=\; \int\limits_0^\pi \sin^2\theta \times \sin\theta\, d\theta \times \int\limits_0^{2\pi} d\phi \;=\; \frac{4}{3} \times 2\pi \;=\; \frac{8\pi}{3} \tag{S.104}$$

and therefore

$$P_{\text{net}} \;=\; \oiint\limits_{4\pi} \frac{dP}{d\Omega}\, d\Omega \;=\; \frac{|A|^2}{2Z_0} \times \frac{8\pi}{3}\,. \tag{S.105}$$