PHY–352 L. Solutions for problem set #4.

Problem **1**:

(a) In a conducting material, the bulk charge density decays exponentially with time t (by flowing to the conductor's boundary) as

$$\rho(\mathbf{r},t) \;=\; \rho_0(\mathbf{r})\times e^{-t/\tau} \quad \text{for } \tau \;=\; \frac{\epsilon\epsilon_0}{\sigma}\,. \tag{S.1}$$

Note: the better the conductor, the faster the bulk charge decay rate. In particular, for the pure silicon with $\sigma = 4.3\cdot 10^{-4}$ ℧/m and $\epsilon = 11.7$,

$$\tau \;=\; \frac{11.7\times 8.85\cdot 10^{-12}\ \mathrm{F/m}}{4.3\cdot 10^{-4}\ \mho/\mathrm{m}} \;=\; 2.4\cdot 10^{-7}\ \mathrm{F}\cdot\Omega \;=\; 2.4\cdot 10^{-7}\ \mathrm{s}. \tag{S.2}$$

(b) Since the high-frequency current flows only through the surface layers — the "skin" — of a wire, the golden upper layers should have thickness of about the skin depth δ, or perhaps to a few times δ. Anything thicker than that would be a waste of gold.

The gold is non-magnetic ($\mu\approx 1$) and has conductivity $\sigma = 4.1\cdot 10^7$ ℧/m, so at frequency $\omega = 2\pi\times 10$ GHz, the skin depth in gold is

$$\delta \;=\; \sqrt{\frac{2}{\mu\mu_0\omega\sigma}} \;=\; \sqrt{\frac{2}{(4\pi\cdot 10^{-7}\ \mathrm{H/m})\times(2\pi\times 10^{10}\ \mathrm{s}^{-1})\times(4.1\cdot 10^{7}\ \mho/\mathrm{m})}} \;\approx\; 0.8\ \mu\mathrm{m}. \tag{S.3}$$

Thus, the wires should be covered in gold to a thickness of 0.8 micron; perhaps a few times that to avoid any currents in the base metal core of the wire, but anything more than a few microns would be a waste of gold.

Problem **2**:

(a) At low or radio frequencies water has dielectric constant $\epsilon \approx 80$. Consequently, at frequency $\omega = 2\pi \times 1$ MHz,

$$-\operatorname{Im}(\sigma_{\text{eff}}(\omega_1)) \;=\; \omega_1 \epsilon\epsilon_0 \;\approx\; 4.5 \cdot 10^{-3}\ \mho/\mathrm{m} \tag{S.4}$$

while at $\omega_2 = 2\pi \times 100$ MHz

$$-\operatorname{Im}(\sigma_{\text{eff}}(\omega_2)) \;=\; \omega_2 \epsilon\epsilon_0 \;\approx\; 0.45\ \mho/\mathrm{m}. \tag{S.5}$$

Comparing these imaginary conductivities due to the displacement current to the real conductivity of water in Lake Austin $\sigma = 0.05\ \mho/\mathrm{m}$, we see that

$$\omega_1 \epsilon\epsilon_0 \;\ll\; \sigma \;\ll\; \omega_2 \epsilon\epsilon_0\,. \tag{S.6}$$

Thus, at 1 MHz the conduction current in water is stronger than the displacement current, but at 100 MHz its the displacement current that's stronger than the conduction current. Consequently, for the purpose of calculating the attenuation rate of radio wave in water, we may use the good-conductor approximation for the 1 MHz radio wave but a poor-conductor approximation for the 100 MHz radio wave.

(b) In the good-conductor approximation

$$\operatorname{Re}(\epsilon_{\text{eff}}) \;=\; \epsilon \;\ll\; \frac{\sigma}{\omega\epsilon_0} \;=\; \operatorname{Im}(\epsilon_{\text{eff}}) \tag{S.7}$$

hence

$$\operatorname{Re}(n) \;\approx\; \operatorname{Im}(n) \;\approx\; \sqrt{\frac{\sigma}{\omega\epsilon_0}} \tag{S.8}$$

and therefore the attenuation rate

$$\alpha \;=\; 2\operatorname{Im}(k) \;\approx\; 2\operatorname{Re}(k) \;=\; \frac{4\pi}{\lambda} \;\approx\; \frac{2}{\delta} \tag{S.9}$$

where δ is the skin depth. Thus, for the water at 1 MHz,

$$\alpha \;=\; \frac{2}{\delta} \;=\; \sqrt{2\mu_0\omega\sigma} \;=\; \sqrt{2(4\pi\cdot 10^{-7}\ \mathrm{H/m})\times(2\pi\times 10^{6}\ \mathrm{s}^{-2})\times(0.05\ \mho/\mathrm{m})} \;\approx\; 0.9\ \mathrm{m}^{-1}. \tag{S.10}$$

OOH, in the poor conductor approximation

$$\mathrm{Re}(\epsilon_{\mathrm{eff}}) \;=\; \epsilon \;\gg\; \frac{\sigma}{\omega\epsilon_0} \;=\; \mathrm{Im}(\epsilon_{\mathrm{eff}}), \tag{S.11}$$

hence (for a non-magnetic material)

$$\mathrm{Im}(n) \;\approx\; \frac{\sigma/\omega\epsilon_0}{2\sqrt{\epsilon}} \tag{S.12}$$

and therefore the attenuation rate

$$\alpha \;=\; \frac{2\omega}{c}\,\mathrm{Im}(n) \;\approx\; \frac{\sigma}{c\epsilon_0\sqrt{\epsilon}} \;=\; \frac{\sigma Z_0}{\sqrt{\epsilon}}\,. \tag{S.13}$$

Note: this attenuation rate does not depend on the frequency (as long as we are in the poor conductor regime). Thus, for high enough frequency of the radio wave (and 100 MHz is high enough for the Lake Austin water), the attenuation rate in the lake becomes

$$\alpha \;=\; \frac{(0.05\ \mho/\mathrm{m})\times(377\ \Omega)}{\sqrt{80}} \;\approx\; 2.1\ \mathrm{m}^{-1}. \tag{S.14}$$

Problem **3**:

The electric and the magnetic fields of an attenuating wave are related to the field of the ordinary plane wave by simple analytic continuation from a real dielectric constant ϵ to a complex

$$\epsilon_{\mathrm{eff}} \;=\; \epsilon \;+\; \frac{i\sigma}{\omega\epsilon_0}\,, \tag{S.15}$$

hence complex refraction index $n = \sqrt{\mu\epsilon_{\mathrm{eff}}}$, complex wave vector $\mathbf{k} = n(\omega/c)\hat{\mathbf{k}}$, and complex wave impedance

$$Z\;=\;\sqrt{\frac{\mu\mu_0}{\epsilon_{\text{eff}}\epsilon_0}}\;=\;\frac{\mu}{n}Z_0\,. \tag{S.16}$$

In terms of these complex parameters, the attenuating EM wave has

$$\begin{aligned}\mathbf{E}(\mathbf{r},t)\;&=\;\mathrm{Re}\left(\vec{\mathcal{E}}\exp\Big((\omega/c)(i\,\mathrm{Re}(n)-\mathrm{Im}(n))\hat{\mathbf{k}}\cdot\mathbf{r}\;-\;i\omega t\Big)\right),\\ \mathbf{H}(\mathbf{r},t)\;&=\;\mathrm{Re}\left(\vec{\mathcal{H}}\exp\Big((\omega/c)(i\,\mathrm{Re}(n)-\mathrm{Im}(n))\hat{\mathbf{k}}\cdot\mathbf{r}\;-\;i\omega t\Big)\right),\end{aligned} \tag{S.17}$$

where the electric and the magnetic amplitudes are related as

$$\vec{\mathcal{H}}\;=\;\frac{1}{Z}\,\hat{\mathbf{k}}\times\vec{\mathcal{E}}. \tag{S.18}$$

For a real wave impedance, the electric and the magnetic amplitudes (S.18) have the same phase, so the electric and the magnetic fields (S.17) oscillate with time at the same phase. But for a complex wave impedance Z there is a phase difference: the amplitudes have different phases according to

$$\mathrm{phase}(\vec{\mathcal{H}})\;=\;\mathrm{phase}(\vec{\mathcal{E}})\;-\;\mathrm{phase}(Z), \tag{S.19}$$

so at any space location $\mathbf{r}$ where the EM fields oscillate with time as

$$\begin{aligned}\mathbf{E}(t)\;&=\;\mathbf{E}_0\cos(\omega t-\phi_e)\;=\;\mathrm{Re}\Big(\mathbf{E}_0e^{i\phi_e}\,e^{-i\omega t}\Big),\\ \mathbf{H}(t)\;&=\;\mathbf{H}_0\cos(\omega t-\phi_h)\;=\;\mathrm{Re}\Big(\mathbf{H}_0e^{i\phi_h}\,e^{-i\omega t}\Big),\end{aligned} \tag{S.20}$$

their phases differ by

$$\phi_h\;-\;\phi_e\;=\;\mathrm{phase}(\vec{\mathcal{H}})\;-\;\mathrm{phase}(\vec{\mathcal{E}})\;=\;-\,\mathrm{phase}(Z). \tag{S.21}$$

Moreover, in light of eq. (S.16),

$$-\,\mathrm{phase}(Z)\;=\;+\,\mathrm{phase}(n)\;=\;+\tfrac{1}{2}\,\mathrm{phase}(\epsilon_{\text{eff}}) \tag{S.22}$$

and hence

$$-\,\mathrm{phase}(Z)\;=\;+\frac{1}{2}\arctan\frac{\sigma}{\omega\epsilon\epsilon_0}\,. \tag{S.23}$$

For a good conductor $\sigma\gg\omega\epsilon\epsilon_0$, hence

$$-\,\mathrm{phase}(Z)\;\approx\;+\frac{\pi}{4}\;=\;+45^\circ, \tag{S.24}$$

so according to eq. (S.21), the magnetic field oscillates with a phase that's 45° ahead of the electric field's.

Next, consider the electric and the magnetic energy densities in the attenuating wave,

$$u_{\rm el}\;=\;\tfrac{1}{2}\epsilon\epsilon_0\mathbf{E}^2,\qquad u_{\rm mag}\;=\;\tfrac{1}{2}\mu\mu_0\mathbf{H}^2. \tag{S.25}$$

Note that the ϵ in the electric equation is the real dielectric constant rather than the complex $\epsilon_{\rm eff}$. Consequently, after time averaging

$$\langle u_{\rm el}\rangle\;=\;\tfrac{1}{4}\epsilon\epsilon_0\,|\vec{\mathcal{E}}|^2\,\exp(-\alpha\hat{\mathbf{k}}\cdot\mathbf{r}),\qquad\langle u_{\rm mag}\rangle\;=\;\tfrac{1}{4}\mu\mu_0\,|\vec{\mathcal{E}}|^2\,\exp(-\alpha\hat{\mathbf{k}}\cdot\mathbf{r}), \tag{S.26}$$

hence the ratio of magnetic energy to electric energy is

$$\mathcal{R}\;\stackrel{\rm def}{=}\;\frac{\langle u_{\rm mag}\rangle}{\langle u_{\rm el}\rangle}\;=\;\frac{\mu\mu_0}{\epsilon\epsilon_0}\times\frac{|\vec{\mathcal{H}}|^2}{|\vec{\mathcal{E}}|^2}\;=\;\frac{\mu\mu_0}{\epsilon\epsilon_0}\times\frac{1}{|Z|^2}\,. \tag{S.27}$$

But according to eq. (S.16), the wave impedance Z is related to the complex $\epsilon_{\rm eff}$ rather then the real dielectric constant, hence

$$\mathcal{R}\;=\;\frac{\mu\mu_0}{\epsilon\epsilon_0}\times\frac{|\epsilon_{\rm eff}|\epsilon_0}{\mu\mu_0}\;=\;\frac{|\epsilon_{\rm eff}|}{\epsilon=\mathrm{Re}(\epsilon_{\rm eff})}\;>\;1. \tag{S.28}$$

In particular, for a good conductor

$$\mathrm{Im}(\epsilon_{\rm eff})\;=\;\frac{\sigma}{\omega\epsilon_0}\;\gg\;\epsilon\;=\;\mathrm{Re}(\epsilon_{\rm eff}), \tag{S.29}$$

hence

$$|\epsilon_{\text{eff}}| \;\approx\; \operatorname{Im}(\epsilon_{\text{eff}}) \;=\; \frac{\sigma}{\omega\epsilon_0} \tag{S.30}$$

and therefore

$$\mathcal{R} \;\approx\; \frac{\sigma}{\omega\epsilon\epsilon_0} \;\gg\; 1. \tag{S.31}$$

Problem **4**:

We saw in class — *cf.* my notes on attenuated waves and skin effect, eq. (71) on page 12, — that the reflectivity off a surface of a good conductor may be approximated as

$$R \;\approx\; 1 \;-\; \frac{4\pi\mu\delta}{\lambda_{\text{vac}}} \tag{S.32}$$

where λ_{vac} is the vacuum wavelength of the EM wave and δ is the skin depth in the reflecting material at the frequency of the wave. The copper metal is non-magnetic, $\mu \approx 1$, so the measured value of the reflectivity $R = 0.85$ for the orange light with $\lambda_{\text{vac}} = 600$ nm translates to the skin depth (at $\omega = (2\pi c/\lambda_{\text{vac}}) = 3.14 \cdot 10^{15}\ \text{s}^{-1}$) being

$$\delta \;=\; \frac{\lambda_{\text{vac}}}{4\pi} \times (1-R) \;\approx\; 7.16\ \text{nm}. \tag{S.33}$$

At the same time, the skin depth is related to the frequency and the conductivity as

$$\delta \;=\; \sqrt{\frac{2}{\mu_0\omega\sigma}}\,, \tag{S.34}$$

(for a non-magnetic material), so the skin depth (S.33) implies high-frequency conductivity

$$\sigma_{\text{HF}} \;=\; \frac{2}{\mu_0\omega\delta^2} \;\approx\; \frac{2}{(4\pi\cdot 10^{-7}\ \text{H/m}) \times (3.14\cdot 10^{15}\ \text{s}^{-1}) \times (7.16\cdot 10^{-9}\ \text{m})^2} \;\approx\; 9.9\cdot 16^{6}\ \mho/\text{m}. \tag{S.35}$$

This is about 6 times less than the DC conductivity of the copper metal.

Problem **5**:

(a) For the deep-water dispersion relation (2), the phase velocity of the wave is

$$v_{\text{phase}} \stackrel{\text{def}}{=} \frac{\omega}{k} = \sqrt{\frac{g}{k}} = \sqrt{g\lambda/2\pi}, \tag{S.36}$$

while the group velocity is

$$v_{\text{group}} \stackrel{\text{def}}{=} \frac{d\omega}{dk} = \frac{\sqrt{g}}{2\sqrt{k}}\,. \tag{S.37}$$

By inspection,

$$v_{\text{group}} = \tfrac{1}{2} v_{\text{phase}}\,, \tag{S.38}$$

quod erat demonstrandum.

(b) As we saw in class — *cf.* my notes on dispersion, especially eqs. (40–47) on pages 9–10, — the dispersion limits the maximal pulse rate that can travel distance L without merging to

$$\nu_{\max} = \sqrt{\frac{v_{\text{group}}^3}{L \times |\omega''|}} \tag{notes.44}$$

where $\omega'' = d^2\omega/dk^2$. For the deep-water dispersion relation (2),

$$\omega'' = -\frac{\sqrt{g}}{4k^{3/2}}\,, \tag{S.39}$$

hence in light of eq. (S.37) for the group velocity

$$\frac{v_{\text{group}}^3}{L \times |\omega''|} = \frac{g}{2L}\,, \tag{S.40}$$

so the maximal signal rate transmitted by the wave on the surface of deep water is

$$\nu_{\max} = \sqrt{\frac{g}{2L}}\,. \tag{S.41}$$

Problem **6**:

In terms of the angular frequency ω and the wave number k, eq. (3) means $\omega = E/\hbar$ and $k = p/\hbar$, so the free particle energy $E = p^2/2m$ translates into the dispersion relation

$$\omega(k) \; = \; \frac{\hbar}{2m} k^2. \tag{S.42}$$

(a) In light of the dispersion relation (S.42), the phase velocity of the de Broglie wave (3) is

$$v_{\text{phase}} \; = \; \frac{\omega}{k} \; = \; \frac{\hbar}{2m} \times k, \tag{S.43}$$

while the group velocity is

$$v_{\text{group}} \; = \; \frac{d\omega}{dk} \; = \; \frac{\hbar}{2m} \times 2k. \tag{S.44}$$

As promised, the group velocity is indeed twice the phase velocity of the de Broglie wave.

(b) Identifying $\hbar \times k_{\text{wave}}$ with the momentum p of the particle, we find that

$$v_{\text{phase}} \; = \; \frac{p}{2m} \quad \text{while} \quad v_{\text{group}} \; = \; \frac{p}{m}\,. \tag{S.45}$$

On the other hand, the classical velocity of the moving particle is $v_{\text{cl}} = p/m$. Thus, we see that the classical particle velocity agrees with the *group velocity* of the de Broglie wave rather than its phase velocity.

In quantum mechanics, this agreement stems from treating the moving quantum particle as a wave packet rather than a perfect plane wave of definite momentum $p = \hbar k$. This way, the particle's wave function has a meaningful expectation value $x_0 = \langle\psi|\,\hat{x}\,|\psi\rangle$ of the location, and the probability distribution has a finite with Δx. As the wave function evolves with time, the center of the wave packet moves with the group velocity of the wave. Classically, this corresponds to the particle's motion, and that's why the classical particle velocity is the group velocity of the de Broglie wave.

Problem **7**:

Given eq. (4) for the gas's dielectric constant, we have

$$2n \times \frac{dn}{d\omega} = \frac{dn^2}{d\omega} = 0 + \frac{Ne^2}{\epsilon_0 m_e} \sum_i \frac{2\omega f_i}{(\omega_i^2 - \omega^2)^2}, \tag{S.46}$$

where every term in the sum is positive regardless of the sign of $\omega_i - \omega$ since $f_i > 0$. Thus, whenever eq. (4) is a valid approximation, we have positive $dn/d\omega$, thus normal dispersion (5).

Next, given eq. (4) we also have

$$\begin{aligned} n^2 - 1 + \frac{\omega}{2}\frac{dn^2}{d\omega} &= \frac{Ne^2}{\epsilon_0 m_e} \sum_i \frac{f_i}{\omega_i^2 - \omega^2} + \frac{Ne^2}{\epsilon_0 m_e} \sum_i \frac{\omega^2 \times f_i}{(\omega_i^2 - \omega^2)^2} \\ &= \frac{Ne^2}{\epsilon_0 m_e} \sum_i \frac{\omega_i^2 \times f_i}{(\omega_i^2 - \omega^2)^2} \end{aligned} \tag{S.47}$$

where every term is positive, hence

$$n^2 + \frac{\omega}{2}\frac{dn^2}{d\omega} > 1. \tag{6}$$

Having derived inequalities (5) and (6), let's apply them to the phase velocity and the group velocity of the EM waves in the gas. In terms of the refraction index n,

$$\begin{aligned} \frac{c}{v_{\text{phase}}} &= n, \\ \frac{c}{v_{\text{group}}} &= n + \omega \times \frac{dn}{d\omega}, \end{aligned} \tag{S.48}$$

hence the normal-dispersion inequality (5) translates to

$$\frac{c}{v_{\text{group}}} - \frac{c}{v_{\text{phase}}} = \omega \times \frac{dn}{d\omega} > 0 \tag{S.49}$$

and therefore $v_{\text{group}} < v_{\text{phase}}$, precisely as in the first eq. (7).

Next,

$$\frac{c^2}{v_{\text{phase}} \times v_{\text{group}}} = n \times \left(n + \omega \frac{dn}{d\omega}\right) = n^2 + \frac{\omega}{2} \times \frac{dn^2}{d\omega} > 1 \quad \langle\!\langle\, \text{by eq. (6)} \,\rangle\!\rangle, \tag{S.50}$$

hence

$$v_{\text{group}} \times v_{\text{phase}} < c^2, \tag{S.51}$$

precisely as in the second eq. (7).

Finally, the two eqs. (7) together immediately lead to

$$v_{\text{group}} < c. \tag{8}$$

Thus, in the normal dispersion regime, the EM waves cannot carry energy or information faster than light.

Of course, in the anomalous dispersion regime the EM waves also cannot carry energy or information faster then light, but proving that becomes a lot more complicated.

Problem **8**:

(a) Let the whole electron ball be displaced as a whole by $\mathbf{d}$, so the ball's center is at distance $\mathbf{d}$ from the proton. But assume $|\mathbf{d} < a$, so the proton remains within the electron ball. Consequently, the electric field of the electron ball on the proton is

$$\mathbf{E} = \frac{(-e)(-\mathbf{d})}{4\pi\epsilon_0\, a^3} \tag{S.52}$$

By Newton's third law, the force on the electron ball is minus the force on the proton, thus

$$\mathbf{F} = -e\mathbf{E} = -\frac{e^2}{4\pi\epsilon_0\, a^3}\,\mathbf{d}. \tag{S.53}$$

In other words, the toy model of the atom acts as an electron on a 3D spring with a force constant

$$\kappa = \frac{e^2}{4\pi\epsilon_0\, a^3}\,, \tag{S.54}$$

so the electron oscillates at frequency

$$\omega_0 = \sqrt{\frac{\kappa}{m_e}} = \sqrt{\frac{e^2}{4\pi\epsilon_0\, a^3 m_e}}\,. \tag{S.55}$$

(b) For the toy model radius $a = 0.5$ Å $= 5 \cdot 10^{-11}$ m, eq. (S.55) yields $\omega_0 \approx 4.5 \cdot 10^{16}\ \mathrm{s}^{-1}$. In other words, the resonance frequency is $\omega_0 = 2\pi \times 7.16$ PHz, which corresponds to the vacuum wavelength $\lambda_0 = 41.8$ nm. This frequency lies in the extreme ultraviolet part of the electromagnetic spectrum.

Preamble to part (c):
Consider eq. (4) for a gas at frequencies much lower than all the resonances ω_i. Then we may approximate

$$\frac{1}{\omega_i^2 - \omega^2} \approx \frac{1}{\omega_i^2} + \frac{\omega^2}{\omega_i^4}, \tag{S.56}$$

hence eq. (4) becomes

$$n^2(\omega) \approx 1 + \frac{Ne^2}{\epsilon_0 m_e}\left(\sum_i \frac{f_i}{\omega_i^2} + \omega^2 \times \sum_i \frac{f_i}{\omega_i^4}\right). \tag{S.57}$$

Furthermore, for a dilute gas $n^2 - 1 \ll 1$ so we may further approximate $n - 1 \approx \frac{1}{2}(n^2 - 1)$, thus

$$\begin{aligned} n(\omega) &\approx 1 + \frac{Ne^2}{2\epsilon_0 m_e}\left(\sum_i \frac{f_i}{\omega_i^2} + \omega^2 \times \sum_i \frac{f_i}{\omega_i^4}\right) && \text{(S.58)} \\ &= 1 + A \times \left(1 + C \times \omega^2\right) && \text{(S.59)} \\ \text{for} \quad A &= \frac{Ne^2}{2\epsilon_0 m_e} \sum_i \frac{f_i}{\omega_i^2} && \text{(S.60)} \end{aligned}$$

$$\text{and}\quad C\ =\ \sum_i\frac{f_i}{\omega_i^4}\Bigg/\sum_i f_i\omega_i^2\,.\text{(S.61)}$$

In terms of the wavelength $\frac{\lambda=2\pi c}{\omega}$, eq. (S.59) becomes the Cauchy formula

$$n(\omega)\ \approx\ 1\ +\ A\times\left(1\ +\ \frac{B}{\lambda^2}\right)\tag{8}$$

where A is as in eq. (S.60) while

$$B\ =\ (2\pi c)^2\times C\ =\ (2\pi c)^2\times\sum_i\frac{f_i}{\omega_i^4}\Bigg/\sum_i f_i\omega_i^2\,.\tag{S.62}$$

(c): For a single resonance model of an atom/molecule, eqs. (S.60) and (S.62) for the Cauchy's coefficients simplify to

$$A\ \approx\ \frac{Ne^2}{2\epsilon_0 m_e}\times\frac{f_0}{\omega_0^2}\tag{S.63}$$

and

$$B\ =\ \frac{(2\pi c)^2}{\omega_0^2}\,.\tag{S.64}$$

In particular, for the resonant frequency obtained in part (b), the model's coefficient of dispersion is

$$B\ =\ \left(\frac{2\pi c}{\omega_0}\right)^2\ =\ \lambda_0^2\ =\ \big(41.8\text{ nm}\big)^2\ \approx 1750\text{ nm}^2\ =\ 1.75\cdot 10^{-15}\text{ m}^2.\tag{S.65}$$

Comparing this value to the measured coefficient of dispersion of the hydrogen gas, $B = 7.7\cdot 10^{-15}\text{ m}^2$, we see that the model is off by a factor of 0.23: it's in the right ballpark but at the wrong end of that ballpark. OOH, getting in the right ballpark — albeit not better than that – with such a primitive model is a cause for celebration rather than regret.

Now consider the coefficient of refraction A. Plugging the resonance frequency (S.55) into eq. (S.63), we get

$$A = Nf_0 \times \frac{e^2}{2\epsilon_0 m_e} \times \left(\frac{1}{\omega_0^2} = \frac{4\pi\epsilon_0 m_e a^3}{e^2} \right) = Nf_0 \times 2\pi a^3. \tag{S.66}$$

At standard conditions, any ideal gas has density

$$N = \frac{P_{\rm std}}{k_{\rm Boltzmann} T_{\rm std}} = 2.687 \cdot 10^{25} \, \frac{\rm molecules}{\rm m^3} \, . \tag{S.67}$$

Also, since the hydrogen molecule H_2 has 2 electrons — one for each hydrogen atom — it has $\sum_i f_i = 2$, so for the single-resonance model we take $f_0 = 2$. Hence,

$$A = 4\pi N a^3 = 4\pi \times (2.687 \cdot 10^{25}\ {\rm m}^{-3}) \times (0.5 \cdot 10^{-10}\ {\rm m})^3 \approx 4.2 \cdot 10^{-5}.{\rm andtherefore}$$

Comparing this number to the measure value of the hydrogen gas refraction coefficient $A_H = 1.36 \cdot 10^{-4}$, we see that the model is off by a factor of of about 3. Again, it's in the right ballpark but not much better than that.