Problem **1**:

(a) For any particular $\mathrm{TE}_{m,n}$ or $\mathrm{TM}_{m,n}$ mode of the vacuum-filled rectangular waveguide, a wave of frequency $\omega = 2\pi f$ may propagate if and only if

$$\omega \ > \ c\Gamma_{m,n} \ = \ c\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}\,, \tag{S.1}$$

or equivalently is and only if

$$\left(\frac{2f}{c}\right)^2 \ > \ \frac{m^2}{a^2} \ + \ \frac{n^2}{b^2}\,. \tag{S.2}$$

For $a = 5.00$ cm, $b = 2.00$ cm, and $f = 10.0$ GHz, hence $(2f/c) = 0.667\ \mathrm{cm}^{-1}$, this inequality becomes

$$0.0400 \times m^2 \ + \ 0.250 \times n^2 \ < \ 0.444, \tag{S.3}$$

which has 6 solutions for $(m,n) \neq (0,0)$ pairs of non-negative integers: $(1,0)$, $(2,0)$, $(1,0)$, $(1,1)$, $(3,0)$, and $(2,1)$. For the solutions with $n = 0$ or $m = 0$ there is only the $\mathrm{TE}_{m,n}$ mode, but for $m > 0$ and $n > 0$ there are both the $\mathrm{TE}_{m,n}$ and the $\mathrm{TM}_{m,n}$ modes.

Altogether, the waveguide has 8 modes which allow propagating 10 GHz waves. In order of their cut-off frequencies $f_{\min} = c\Gamma_{m,n}/2\pi$, they are:

- The $\mathrm{TE}_{1,0}$ mode, $f_{\min} = 3.00$ GHz.
- The $\mathrm{TE}_{2,0}$ mode, $f_{\min} = 6.00$ GHz.
- The $\mathrm{TE}_{0,1}$ mode, $f_{\min} = 7.50$ GHz.
- The $\mathrm{TE}_{1,1}$ and the $\mathrm{TM}_{1,1}$ modes, $f_{\min} = 8.08$ GHz.
- The $\mathrm{TE}_{3,0}$ mode, $f_{\min} = 9.00$ GHz.
- The $\mathrm{TE}_{2,1}$ and the $\mathrm{TM}_{2,1}$ modes, $f_{\min} = 9.60$ GHz.

(b) As we see from the above table, at $f < 3$ GHz no mode can propagate through the waveguide while at $f > 6$ GHz there are at least two modes capable of propagation. OOH, in the frequency range between 3 GHz and 6 GHz, there is only one mode that can propagate through the waveguide, namely the $\mathrm{TE}_{1,0}$ mode.

(c) For any particular mode, and for any frequency ω above the cut-off frequency $\Omega = c\Gamma$, the dispersion relation between the wave number k and the frequency ω is

$$(\omega/c)^2 \;=\; \Gamma^2 \;+\; k^2 \tag{S.4}$$

or equivalently

$$c^2k^2 \;=\; \omega^2 \;-\; \Omega^2. \tag{S.5}$$

Consequently, the group velocity of the wave in the waveguide is

$$v_g \;=\; \frac{d\omega}{dk} \;=\; \frac{d\omega^2}{dk^2} \Big/ \frac{2\omega}{2k} \;=\; \frac{kc^2}{\omega} \;=\; c \times \frac{\sqrt{\omega^2 - \Omega^2}}{\omega}\,. \tag{S.6}$$

of in terms of the cyclic frequencies

$$v_g \;=\; c \times \sqrt{1 \;-\; \frac{f_{\min}^2}{f^2}}\,. \tag{S.7}$$

For the $\mathrm{TE}_{1,0}$ mode of the waveguide in question $f_{\min} = 3$ GHz (*cf.* part (a)) while the frequency of the wave is $f = 5$ GHz, hence $\sqrt{1 - (f_{\min}/f)^2} = 0.8$ and hence the group velocity $v_g = 0.8\,c$.

(d) The maximal pulse rate that can be send down a dispersive communication line without the pulses merging together is

$$\nu_{\max} \;=\; \sqrt{\frac{v_g^3}{|\omega''| \times L}} \tag{S.8}$$

where L is the line's length and

$$\omega'' \;=\; \frac{d^2\omega}{dk^2} \;=\; \frac{dv_g}{dk}\,. \tag{S.9}$$

For a particular wave mode in a waveguide,

$$v_g \;=\; c\times\frac{\sqrt{\omega^2-\Omega^2}}{\omega} \;=\; c\times\frac{k}{\sqrt{k^2+\Gamma^2}}\,, \tag{S.10}$$

hence

$$\omega'' \;=\; c\left(\frac{1}{\sqrt{k^2+\Gamma^2}}\;-\;\frac{k^2}{(k^2+\Gamma^2)^{3/2}}\right) \;=\; \frac{c\Gamma^2}{(k^2+\Gamma^2)^{3/2}}, \tag{S.11}$$

or in frequency terms

$$\omega'' \;=\; \frac{c^2\Omega^2}{\omega^3}\,. \tag{S.12}$$

Consequently,

$$\nu^2_{\rm max} \;=\; \frac{v_g^3}{L\times|\omega''|} \;=\; \frac{c^3(\omega^2-\Omega^2)^{3/2}/\omega^3}{L\times c^2\Omega^2/\omega^3} \;=\; \frac{c\Omega}{L}\times\frac{(\omega^2-\Omega^2)^{3/2}}{\Omega^3}\,, \tag{S.13}$$

or in terms of the cyclic frequencies,

$$\nu^2_{\rm max} \;=\; 2\pi f_{\rm min}\times\frac{c}{L}\times\left(\frac{f^2}{f^2_{\rm min}}-1\right)^{3/2}. \tag{S.14}$$

For the waveguide in question and the $\mathrm{TE}_{1,0}$ mode with $f_{\rm min}=3$ GHz while the wave has frequency $f=5$ GHz, we have

$$2\pi f_{\rm min}\times\frac{c}{L} \;=\; 2\pi\times(3\cdot10^9\ \mathrm{Hz})\times\frac{3\cdot10^8\ \mathrm{m/s}}{10\ \mathrm{m}} \;=\; 56.5\cdot10^{16}\ \mathrm{Hz}^2 \tag{S.15}$$

while

$$\left(\frac{f^2}{f^2_{\rm min}}-1\right)^{3/2} \;=\; \left(\tfrac{25}{9}-1\right)^{3/2} \;=\; \frac{64}{27} \;\approx\; 2.37, \tag{S.16}$$

so the maximal pulse rate is

$$\nu_{\rm max} \;=\; \sqrt{\left(56.5\cdot10^{16}\ \mathrm{Hz}^2\right)\times2.37} \;\approx\; 1.16\cdot10^9\,\frac{\mathrm{pulses}}{\mathrm{second}}\,. \tag{S.17}$$

Problem **2**:

(a) Let's start with the time-averaged energy density

$$\langle u\rangle \;=\; \frac{\mu_0}{4}\left|\hat{\mathbf{H}}\right|^2 \;+\; \frac{\epsilon_0}{4}\left|\hat{\mathbf{E}}\right|^2. \tag{S.18}$$

For the wave in question

$$\left|\hat{\mathbf{H}}\right|^2 \;=\; |H_0|^2\times\left(\cos^2\frac{\pi x}{a} \;+\; \frac{k^2a^2}{\pi^2}\times\sin^2\frac{\pi x}{a}\right), \tag{S.19}$$

$$\left|\hat{\mathbf{E}}\right|^2 \;=\; \frac{\mu_0}{\epsilon_0}\,|H_0|^2\times\frac{\omega^2a^2}{\pi^2c^2}\times\sin^2\frac{\pi x}{a}\,, \tag{S.20}$$

hence

$$\langle u\rangle \;=\; \frac{\mu_0}{4}\,|H_0|^2\times\left(\cos^2\frac{\pi x}{a} \;+\; \frac{a^2}{\pi^2}\left(\frac{\omega^2}{c^2}+k^2\right)\times\sin^2\frac{\pi x}{a}\right) \tag{S.21}$$

Moreover, for the $\mathrm{TE}_{1,0}$ mode in question

$$(\omega/c)^2 \;=\; k^2 \;+\; \Gamma^2 \quad\text{for}\quad \Gamma \;=\; \frac{\pi}{a}\,, \tag{S.22}$$

hence

$$\frac{a^2}{\pi^2}\left(\frac{\omega^2}{c^2}+k^2\right) \;=\; \frac{2k^2+\Gamma^2}{\Gamma^2} \tag{S.23}$$

and therefore

$$\langle u\rangle \;=\; \frac{\mu_0}{4}\,|H_0|^2\times\left(\cos^2\frac{\pi x}{a} \;+\; \frac{2k^2+\Gamma^2}{\Gamma^2}\times\sin^2\frac{\pi x}{a}\right). \tag{S.24}$$

Next, the (time-averaged) Poynting vector

$$\langle\mathbf{S}\rangle \;=\; \tfrac{1}{2}\,\mathrm{Re}\big(\hat{\mathbf{E}}^*\times\hat{\mathbf{H}}\big). \tag{S.25}$$

In light of eqs. (1) — and in particularly $H_y=E_x=E_z=0$, — we have

$$\begin{aligned}\langle S_x\rangle \;&=\; \tfrac{1}{2}\,\mathrm{Re}\big(E_y^*\,H_z\big)\\ &=\; \tfrac{1}{2}\,\mathrm{Re}\left(\frac{-i\omega\mu_0 a}{\pi}\right)|H_0|^2\,\sin\frac{\pi x}{a}\cos\frac{\pi x}{a}\\ &=\; 0,\end{aligned} \tag{S.26}$$

$$\begin{aligned} \langle S_y\rangle &= 0, && \text{(S.27)}\\ \langle S_z\rangle &= -\tfrac{1}{2}\,\mathrm{Re}\big(\hat{E}_y^*\,H_x\big)\\ &= +\tfrac{1}{2}\,\mathrm{Re}\left(\frac{-i\omega\mu_0 a}{\pi}\times\frac{-ika}{\pi}\right)|H_0|^2\,\sin^2\frac{\pi x}{a}\\ &= +\frac{\mu_0}{2}\,|H_0|^2\times\frac{\omega k}{(\pi/a)^2}\times\sin^2\frac{\pi x}{a}\,. && \text{(S.28)} \end{aligned}$$

Note: whole the instantaneous Poynting vector $\mathbf{S}$ has both x and z components, the time-averaged x component vanishes, so the time-averaged power density flows in z direction down the waveguide.

Now, let's integrate the energy density and the Poynting vector over the waveguide's cross-section $a \times b$. Since neither $\langle u\rangle$ nor $\langle S_z\rangle$ depend on the y coordinate, integrating over dy simply yields an overall factor b. As to the x coordinate, various terms in $\langle u\rangle$ and $\langle S_z\rangle$ behave as $\cos^2(\pi x/a)$ or $\sin^2(\pi x/a)$, and both of these integrate to

$$\int_0^a dx\,\cos^2\frac{\pi x}{a} = \int_0^a dx\,\sin^2\frac{\pi x}{a} = \frac{a}{2}\,.$$

Consequently,

$$\frac{\text{energy}}{\text{length}} = \iint dx\,dy\,\langle u\rangle = \frac{\mu_0|H_0|^2}{4}\times\frac{ab}{2}\times\left(1+\frac{2k^2+\Gamma^2}{\Gamma^2}\right) = \frac{ab\mu_0|H_0|^2}{8}\times\frac{2(k^2+\Gamma^2)}{\Gamma^2}\,. \tag{S.29}$$

Likewise,

$$\begin{pmatrix}\text{wave}\\ \text{power}\end{pmatrix} = \iint dx\,dy\,\langle S_z\rangle = \frac{\mu_0|H_0|^2}{2}\times\frac{ab}{2}\times\frac{\omega k}{(\pi/a)^2} = \frac{ab\mu_0|H_0|^2}{4}\times\frac{\omega k}{\Gamma^2}\,. \tag{S.30}$$

(b) The velocity of the EM energy in the waveguide obtains as the ration of the net (time-averaged) power flowing down the waveguide to the average energy per unit of the waveguide's length. In light of eqs. (S.30) and (S.29) from part (a), we have

$$\begin{aligned}
v_{\text{energy}} &= \frac{\text{power}}{\text{energy/length}} \\
&= \frac{ab\mu_0|H_0|^2}{4} \times \frac{\omega k}{\Gamma^2} \Big/ \frac{ab\mu_0|H_0|^2}{8} \times \frac{2(k^2+\Gamma^2)}{\Gamma^2} \\
&= \frac{\omega k}{(k^2+\Gamma^2)} \\
&\quad \langle\!\langle \text{ using } k^2+\Gamma^2 = (\omega/c)^2 \,\rangle\!\rangle \\
&= \frac{c^2 k}{\omega}\,.
\end{aligned} \tag{S.31}$$

But back in eq. (S.6) from problem 1(b), we saw that the group velocity of the wave in the waveguide is

$$v_{\text{group}} = \frac{c^2 k}{\omega}\,, \tag{S.32}$$

so comparing this formula to eq. (S.31) we immediately see that the energy velocity of the EM wave in a waveguide is precisely equal to the group velocity of the wave.

<u>Problem **3**</u>:

(a) First, the divergence equations:

$$\nabla\cdot\mathbf{E} = Ae^{ikz-i\omega t}\left(ik\hat{\mathbf{z}}\cdot\frac{\hat{\mathbf{s}}}{s} + \nabla\cdot\frac{\hat{\mathbf{s}}}{s}\right) = 0 \tag{S.33}$$

because $\hat{\mathbf{z}}\cdot\hat{\mathbf{s}} = 0$ and $\nabla\cdot(\hat{\mathbf{s}}/s) = 0$. Likewise,

$$\nabla\cdot\mathbf{B} = \frac{A}{c}e^{ikz-i\omega t}\left(ik\hat{\mathbf{z}}\cdot\frac{\hat{\boldsymbol{\phi}}}{s} + \nabla\cdot\frac{\boldsymbol{\phi}}{s}\right) = 0 \tag{S.34}$$

because $\hat{\mathbf{z}}\cdot\hat{\boldsymbol{\phi}} = 0$ and $\nabla\cdot(\hat{\boldsymbol{\phi}}/s) = 0$.

Next, the Induction Law:

$$\begin{aligned}
\nabla\times\mathbf{E} \;&=\; Ae^{ikz-i\omega t}\left(ik\hat{\mathbf{z}}\times\frac{\hat{\mathbf{s}}}{s}\;+\;\nabla\times\frac{\hat{\mathbf{s}}}{s}\right)\\
&=\; Ae^{ikz-i\omega t}\left(ik\frac{\hat{\boldsymbol{\phi}}}{s}\;+\;\mathbf{0}\right)\\
&=\; +i\omega\,\frac{A\sqrt{\epsilon}}{c}\,e^{ikz-i\omega t}\,\frac{\hat{\boldsymbol{\phi}}}{s}\\
&=\; -\frac{\partial\mathbf{B}}{\partial t}\,.
\end{aligned} \tag{S.35}$$

Finally, the Maxwell-Ampere Law (without the conduction current):

$$\begin{aligned}
\nabla\times\mathbf{B} \;&=\; \frac{A\sqrt{\epsilon}}{c}\,e^{ikz-i\omega t}\left(ik\hat{\mathbf{z}}\times\frac{\hat{\boldsymbol{\phi}}}{s}\;+\;\nabla\times\frac{\hat{\boldsymbol{\phi}}}{s}\right)\\
&=\; \frac{A\sqrt{\epsilon}}{c}\,e^{ikz-i\omega t}\left(ik\frac{(-\hat{\mathbf{s}})}{s}\;+\;\mathbf{0}\right)\\
&=\; \frac{\epsilon}{c^2}\,(-i\omega)\,A\,e^{ikz-i\omega t}\,\frac{\hat{\mathbf{s}}}{s}\\
&=\; \frac{\epsilon}{c^2}\,(-i\omega)\mathbf{E}\;=\;\frac{\epsilon}{c^2}\,\frac{\partial\mathbf{E}}{\partial t}\\
&=\; \mu_0\frac{\partial\mathbf{D}}{\partial t}\,.
\end{aligned} \tag{S.36}$$

This completes verifying the Maxwell equations for the EM fields (2).

(b) The voltage between the wires obtains as

$$V(z,t)\;=\;\int_a^b ds\,E_s(s,z,t)\;=\;A\,e^{ikz-i\omega t}*\int_a^b\frac{ds}{s}\;=\;A\,e^{ikz-i\omega t}*\ln\frac{b}{a}\,. \tag{S.37}$$

In other words,

$$V(z,t)\;=\;\hat{V}e^{ikz-i\omega t}\quad\text{for the amplitude }\hat{V}\;=\;A\ln\frac{b}{a}\,. \tag{S.38}$$

The current in the inner wire obtains from the Ampere's Law:

$$\mu_0 I(z,t)\;=\;\oint B_\phi(s,z,t)\,s\,d\phi\;=\;\frac{A\sqrt{\epsilon}}{c}\,e^{ikz-i\omega t}*2\pi. \tag{S.39}$$

Or in other words,

$$I(z,t) \;=\; \hat{I}e^{ikz-i\omega t} \quad \text{for the amplitude } \hat{I} \;=\; \frac{2\pi A\sqrt{\epsilon}}{c\mu_0}\,. \tag{S.40}$$

Taking the ratio of the voltage and the current amplitudes, we get the cable's impedance as

$$Z \;=\; \frac{\hat{V}}{\hat{I}} \;=\; \frac{c\mu_0}{\sqrt{\epsilon}} \times \frac{\ln(b/a)}{2\pi} \;=\; \frac{Z_0}{\sqrt{\epsilon}} \times \frac{\ln(b/a)}{2\pi}\,. \tag{S.41}$$

for $Z_0 = c\mu_0 \approx 377\ \Omega$ being the wave impedance of the free space.

(c) In light of eq. (S.41), we want

$$\frac{Z_0}{\sqrt{\epsilon}} \times \frac{\ln(b/a)}{2\pi} \;=\; Z_{\text{given}} \;=\; 75\ \Omega. \tag{S.42}$$

Therefore, we need

$$\frac{\ln(b/a)}{2\pi} \;=\; \frac{Z_{\text{given}}}{Z_0} \times \sqrt{\epsilon} \;=\; \frac{75\ \Omega}{377\Omega} \times \sqrt{2.25} \;\approx\; 0.298 \tag{S.43}$$

and hence

$$\frac{b}{a} \;\approx\; \exp(2\pi \times 0.298) \;\approx\; 6.52, \tag{S.44}$$

so if the radius of the inner wire is $a = 1.00$ mm, then the inner radius of the outer wire should be $b \approx 6.52$ mm.

Problem **4**:

(a) Within the range $|x| \leq ct$,

$$\frac{\partial f}{\partial t} = 4c(ct-|x|)^3, \qquad \frac{\partial f}{\partial x} = -4(ct-|x|)^3\,\mathrm{sign}(x). \tag{S.45}$$

Consequently, the electric field is

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t} = \mathbf{0} + \frac{N}{4c}\frac{\partial f}{\partial t}\,\hat{\mathbf{y}} = N(ct-|x|)^3\hat{\mathbf{y}}, \tag{S.46}$$

while the magnetic field is

$$\mathbf{B} = \nabla\times\mathbf{A} = -\frac{N}{4c}\frac{\partial f}{\partial x}\,\hat{\mathbf{z}} = \frac{N}{c}(ct-|x|)^3\,\mathrm{sign}(x)\,\hat{\mathbf{z}}. \tag{S.47}$$

Outside the range $|x| < ct$, both the electric and the magnetic fields vanish.

Note: at the range boundaries $x = \pm ct$, both the electric and the magnetic fields are continuous. OOH, the magnetic field has a discontinuity at $x = 0$ due to the factor $\mathrm{sign}(x)$ abruptly changing sign.

(b) Let's start with the divergences of the EM fields. Both $\mathbf{E}$ and $\mathbf{B}$ fields depend on x and t but not on y or z; also, $\mathbf{E}$ points in the y direction while $\mathbf{B}$ points in the z direction. This automagically makes both fields divergence-less,

$$\nabla\cdot\mathbf{E} = 0, \qquad \nabla\cdot\mathbf{B} = 0. \tag{S.48}$$

Consequently, both time-independent Maxwell equations are satisfied, provided there are no electric charges at all, $\rho(x,y,z,t) \equiv 0$.

Next, the Induction Law:

$$\nabla\times\mathbf{E} = N\frac{\partial(ct-|x|)^3}{\partial x}\,\hat{\mathbf{x}}\times\hat{\mathbf{y}} = -3N(ct-|x|)^2\,\mathrm{sign}(x)*\hat{\mathbf{z}} \tag{S.49}$$

while

$$\frac{\partial\mathbf{B}}{\partial t} = \frac{N}{c}\frac{\partial(ct-|x|)^3}{\partial t}\,\mathrm{sign}(x)\,\hat{\mathbf{z}} = +3N(ct-|x|)^2\,\mathrm{sign}(x)*\hat{\mathbf{z}}, \tag{S.50}$$

thus by inspection

$$\nabla \times \mathbf{E} \;=\; -\frac{\partial \mathbf{B}}{\partial t}\,. \tag{S.51}$$

Finally, the Maxwell–Ampere Law

$$\mu_0 \mathbf{J} \;=\; \nabla \times \mathbf{B} \;-\; \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}\,. \tag{S.52}$$

For the EM fields (S.46) and (S.47),

$$\begin{aligned} \nabla \times \mathbf{B} \;&=\; \frac{N}{c}\frac{\partial}{\partial x}\Big((ct-|x|)^3\,\mathrm{sign}(x)\Big)\,\hat{\mathbf{x}}\times\hat{\mathbf{z}} \\ &=\; \frac{N}{c}\Big(-3(ct-|x|)^2 \,+\, 2(ct-|x|)^3\delta(x)\Big)(-\hat{\mathbf{y}}) \\ &=\; +3\frac{N}{c}(ct-|x|)^2 * \hat{\mathbf{y}} \;-\; 2Nc^2t^3\delta(x) * \hat{\mathbf{y}}, \end{aligned} \tag{S.53}$$

while

$$\frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} \;=\; \frac{N}{c^2}\frac{\partial(ct-|x|)^3}{\partial t} * \hat{\mathbf{y}} \;=\; +3\frac{N}{c}(ct-|x|)^2 * \hat{\mathbf{y}}. \tag{S.54}$$

Plugging these formulae into the Maxwell–Ampere Law (S.52), we see that they almost cancel except for the δ-function term at $x=0$, thus the MA Law holds for the current

$$\mu_0 \mathbf{J} \;=\; -2Nc^2t^3\delta(x) * \hat{\mathbf{y}}\,. \tag{S.55}$$

Physically, the δ-function here means a 2D current sheet at $x=0$:

$$\mathbf{J}(x,t) \;=\; \mathbf{K}(t)\delta(x) \quad \text{for} \quad \mathbf{K} \;=\; \frac{2Nc^2}{\mu_0}\,t^3 * (-\hat{\mathbf{y}}). \tag{S.56}$$

Problem **5**:

(a) It is easy to see that the vector potential (5) is a gradient,

$$\mathbf{A}(\mathbf{r},t) \;=\; \nabla G(\mathbf{r},t) \quad \text{for} \quad F(t) \;=\; \frac{Qt}{4\pi\epsilon_0\, r}\,. \tag{S.57}$$

Consequently,

$$\mathbf{B} \;=\; \nabla\times\mathbf{A} \;= \nabla\times\nabla(\cdots) \;=\; 0, \tag{S.58}$$

so the magnetic field is identically zero.

As to the electric field,

$$\mathbf{E}(\mathbf{r},t) \;=\; -\nabla V \;-\; \frac{\partial\mathbf{A}}{\partial t} \;=\; \mathbf{r}0 \;+\; \frac{Q}{4\pi\epsilon_0}\,\frac{\mathbf{n}}{r^2}\,. \tag{S.59}$$

Unlike the vector potential, this electric field is time-independent. Moreover, it is precisely the Coulomb field of a static point charge Q at the origin.

(b) A gauge transform changes the vector and the scalar potentials according to

$$\mathbf{A}'(\mathbf{r},t) \;=\; \mathbf{A}(\mathbf{r},t) \;+\; \nabla\Lambda(\mathbf{r},t), \qquad V'(\mathbf{r},t) \;=\; V(\mathbf{r},t) \;-\; \frac{\partial\Lambda(\mathbf{r},t)}{\partial t}\,, \tag{S.60}$$

for some function $\Lambda(\mathbf{r},t)$ of position and time. We saw in eq. (S.57) that $\mathbf{A}(\mathbf{r},t)$ is a pure gradient of some $F(\mathbf{r},t)$, so by letting

$$\Lambda(\mathbf{r},t) \;=\; -F(\mathbf{r},t) \;=\; -\frac{Qt}{4\pi\epsilon_0\, r} \tag{S.61}$$

we may completely gauge-away the vector potential, *i.e.* set $\mathbf{A}'\equiv 0$; indeed,

$$\mathbf{A}'(\mathbf{r},t) \;=\; \bigl(\mathbf{A}(\mathbf{r},t) = \nabla F(\mathbf{r},t)\bigr) \;+\; \nabla\Lambda(\mathbf{r},t) \;\equiv\; 0 \quad \text{for } \Lambda \;\equiv\; -F. \tag{S.62}$$

Of course, the same gauge transform also turns on the scalar potential

$$V'(\mathbf{r},t) \;=\; V(\mathbf{r},t) \;-\; \frac{\partial\Lambda}{\partial t} \;=\; 0 \;+\; \frac{\partial F}{\partial t} \;=\; +\frac{Q}{4\pi\epsilon_0\, r}\,, \tag{S.63}$$

which is precisely the electrostatic Coulomb potential of a point charge Q at the origin.

Problem **6**:

For the potentials (6), the electric field is

$$\mathbf{E} = -\nabla V - \frac{\partial\mathbf{A}}{\partial t} = \mathbf{0} - A_0\omega\,\sin(kx-\omega t)\,\hat{\mathbf{y}}, \tag{S.64}$$

while the magnetic field is

$$\mathbf{B} = \nabla\mathbf{A} = -kA_0\sin(kx-\omega t)\,(\hat{\mathbf{x}}\times\hat{\mathbf{y}}) = -A_0k\sin(kx-\omega t)\,\hat{\mathbf{z}}. \tag{S.65}$$

By inspection, these EM fields describe a plane wave traveling in $+x$ direction and polarized along the y axis. This wave should travel at the speed of light, so we need $k=\omega/c$. We also need both the electric and the magnetic amplitudes to be $\perp$ to the wave direction — which indeed they are — and the relation $c\vec{\mathcal{B}}=\hat{\mathbf{k}}\times\vec{\mathcal{E}}$ between these amplitudes. But according to eqs. (S.64) and (S.65),

$$\vec{\mathcal{E}} = -A_0\omega\hat{\mathbf{y}}, \qquad c\vec{\mathcal{B}} = -A_0ck\,(\hat{\mathbf{x}}=\hat{\mathbf{k}})\times\hat{\mathbf{y}}, \tag{S.66}$$

so for $\omega=ck$ the two amplitudes are indeed related as $c\vec{\mathcal{B}}=\hat{\mathbf{k}}\times\vec{\mathcal{E}}$. Thus, as long as $\omega=ck$, the plane wave (S.64)+(S.65) indeed obeys all the Maxwell equations.

Problem **7**:

Note: for $V\equiv 0$, the Landau gauge condition

$$\nabla\cdot\mathbf{A} + \frac{1}{c^2}\frac{\partial V}{\partial t} \equiv 0 \tag{S.67}$$

is equivalent to the Coulomb gauge condition $\nabla\cdot\mathbf{A}\equiv 0$. For all 3 potentials in question — (3), (5), and (6), — we have $V\equiv 0$, so they either satisfy both Coulomb and Landau gauge conditions or neither condition.

The vector potential (3) points in the y direction but depends only on x and t but not on y or z, so it automagically has zero divergence, $\nabla\cdot\mathbf{A}=0$. Thus, the potentials (3) satisfy both the Coulomb and the Landau gauge conditions.

The vector potential (6) has exactly similar behavior: It in the y direction but depends only on x and t but not on y or z, hence automagically $\nabla \cdot \mathbf{A} = 0$. Thus, the potentials (6) also satisfy both the Coulomb and the Landau gauge conditions.

For the potentials (5), the situation is more subtle. At first blush, one might evaluate

$$\nabla \cdot \frac{\mathbf{n}}{r^2} \;=\; \left(\frac{\partial}{\partial r} + \frac{2}{r}\right)\frac{1}{r^2} \;=\; 0 \tag{S.68}$$

and hence $\nabla \cdot \mathbf{A} = 0$. However, eq. (S.68) is valid only for $r > 0$ but misses a δ-spike at the origin. Indeed,

$$\nabla \cdot \frac{\mathbf{n}}{r^2} \;=\; \nabla^2 \frac{-1}{r} \;=\; +4\pi\delta^{(3)}(\mathbf{r}), \tag{S.69}$$

hence

$$\nabla \cdot \mathbf{A} \;=\; -\frac{Qt}{\epsilon_0}\,\delta^{(3)}(\mathbf{r}) \;\not\equiv\; 0. \tag{S.70}$$

Thus, the potentials (5) obey neither Coulomb nor Landau gauge conditions.

<u>Problem **8**</u>:

(a) In light of

$$\nabla \cdot \frac{\mathbf{n}}{r^2} \;=\; \nabla^2 \frac{-1}{r} \;=\; +4\pi\delta^{(3)}(\mathbf{r}), \tag{S.69}$$

the current density (7) has divergence

$$\nabla \cdot \mathbf{J} \;=\; -\dot{Q}(t) * \delta^{(3)}(\mathbf{r}) \;=\; -\frac{\partial\rho}{\partial t}\,, \tag{S.71}$$

so the the current and charge densities (7) indeed obey the continuity equation.

(b) Let's start with the scalar potential in the Coulomb gauge. The solution to eq. (8) is the instantaneous Coulomb potential of the charge density ρ, hence for the time-dependent point charge $Q(t)$,

$$V(\mathbf{r},t) \;=\; \frac{Q(t)}{4\pi\epsilon_0}\,\frac{1}{r}\,, \tag{S.72}$$

Next, let's evaluate the RHS of eq. (9) for the vector potential:

$$\frac{1}{c^2}\nabla\left(\frac{\partial V}{\partial t}\right) \;=\; \frac{1}{c^2}\nabla\frac{\dot{Q}(t)}{4\pi\epsilon_0\, r} \;=\; \frac{\mu_0\dot{Q}}{4\pi}\,\frac{-\mathbf{n}}{r^2}\,, \tag{S.73}$$

hence

$$\mu_0\mathbf{J}_T \;=\; \mu_0\mathbf{J} \;-\; \frac{1}{c^2}\nabla\left(\frac{\partial V}{\partial t}\right) \;=\; -\frac{\mu_0\dot{Q}}{4\pi}\,\frac{\mathbf{n}}{r^2} \;+\; \frac{\mu_0\dot{Q}}{4\pi}\,\frac{\mathbf{n}}{r^2} \;=\; 0. \tag{S.74}$$

Therefore, eq. (9) for the vector potential in the Coulomb gauge becomes

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} \;-\; \nabla^2\right)\mathbf{A}(\mathbf{r},t) \;=\; \mu_0\mathbf{J}_T(\mathbf{r},t) \;=\; 0, \tag{S.75}$$

and the only causal solution to this equation is $\mathbf{A}\equiv 0$.

(b) Given the scalar potential (S.72) and $\mathbf{A}\equiv 0$, we immediately find that the magnetic field is completely absent, $\mathbf{B}=\nabla\mathbf{A}\equiv 0$, while the electric field is the Coulomb field of the time-dependent point charge,

$$\mathbf{E} \;=\; -\nabla V \;=\; \frac{Q(t)}{4\pi\epsilon_0}\,\frac{\mathbf{n}}{r^2}\,. \tag{S.76}$$

Now let's verify the relevant Maxwell equations for this field. First, the Gauss Law:

$$\nabla\cdot\mathbf{E} \;=\; \frac{Q(t)}{\epsilon_0}\,\delta^{(3)}(\mathbf{r}) \;=\; \rho(\mathbf{r},t). \qquad \langle\!\langle\,\text{Check!}\,\rangle\!\rangle \tag{S.77}$$

Next, the Induction Law, which for $\mathbf{B}\equiv 0$ demands $\nabla\times\mathbf{E}\equiv 0$. And indeed,

$$\nabla\times\mathbf{E} \;=\; \frac{Q(t)}{4\pi\epsilon_0}\,\nabla\times\frac{\mathbf{n}}{r^2} \;=\; 0. \qquad \langle\!\langle\,\text{Check!}\,\rangle\!\rangle \tag{S.78}$$

The $\nabla\cdot\mathbf{B}=0$ is automatic for $\mathbf{B}=0$, while the Maxwell–Ampere Law for $\mathbf{B}=0$ becomes

$$\epsilon_0\,\frac{\partial\mathbf{E}}{\partial t}\;+\;\mathbf{J}\;=\;\nabla\times\mathbf{H}\;=\;0. \tag{S.79}$$

But for the electric field in question,

$$\epsilon_0\,\frac{\partial\mathbf{E}}{\partial t}\;=\;+\frac{\dot{Q}(t)}{4\pi}\,\frac{\mathbf{n}}{r^2}\;=\;-\mathbf{J}[\text{from eq. (7)}], \tag{S.80}$$

so the Maxwell–Ampere equation indeed holds true.

Problem **9**:

(a) There are no electric charges in the system, just the current, so in the Landau gauge $V\equiv 0$. As to the vector potential,

$$\begin{aligned}\mathbf{A}(\mathbf{r},t)\;&=\;\frac{\mu_0}{4\pi}\iiint d^3\mathrm{Vol}'\,\frac{\mathbf{J}(\mathbf{r}',t_{\mathrm{ret}})}{\mathcal{R}}\\ &\to\frac{\mu_0\hat{\mathbf{z}}}{4\pi}\int\limits_{-\infty}^{+\infty}dz'\,\frac{I(t_{\mathrm{ret}})}{\mathcal{R}}\\ &=\;\frac{\mu_0 N\hat{\mathbf{z}}}{4\pi}\int\limits_{-\infty}^{+\infty}dz'\,\frac{\delta(t_{\mathrm{ret}})}{\mathcal{R}}\,,\end{aligned} \tag{S.81}$$

where the δ-spike happens for

$$t_{\mathrm{ret}}\;=\;t\;-\;\frac{\mathcal{R}}{c}\;=\;0. \tag{S.82}$$

In polar coordinates (s,ϕ,z) for the $\mathbf{r}$, this means

$$t\;>\;0\quad\text{and}\quad(ct)^2\;=\;\mathcal{R}^2\;=\;s^2\;+\;(z-z')^2, \tag{S.83}$$

so the integral (S.81) vanishes for $ct<s$. OOH, for $ct>s$, there are two δ-spikes for

$z' = z \pm \sqrt{(ct)^2 - s^2}$, so the integral amounts to

$$\int_{-\infty}^{+\infty} dz' \, \frac{\delta(t_{\rm ret})}{\mathcal{R}} \;=\; 2 \times \frac{1}{\mathcal{R}} \times \frac{1}{|\partial t_{\rm ret}/\partial z'|} \tag{S.84}$$

where the RHS is evaluated for $t_{\rm ret} = 0$. Consequently, $\mathcal{R} \to ct$ while

$$\frac{\partial t_{\rm ret}}{\partial z'} \;=\; \frac{(z - z')}{c^2 t} \;\to\; \pm\frac{\sqrt{(ct)^2 - s^2}}{c^2 t} \tag{S.85}$$

hence

$$\int_{-\infty}^{+\infty} dz' \, \frac{\delta(t_{\rm ret})}{\mathcal{R}} \;=\; 2 \times \frac{1}{ct} \times \frac{c^2 t}{\sqrt{(ct)^2 - s^2}} \;=\; \frac{2c}{\sqrt{(ct)^2 - s^2}}\,, \tag{S.86}$$

and therefore

$$\mathbf{A}(s,t) \;=\; \frac{\mu_0 c N}{2\pi} \, \frac{\hat{\mathbf{z}}}{\sqrt{(ct)^2 - s^2}}\,. \tag{S.87}$$

(b) First of all, for $ct < s$ we have $V = 0$ and $\mathbf{A} = 0$, hence $\mathbf{E} = 0$ and $\mathbf{B} = 0$. Instead, the EM fields show up only for $ct > s$. Specifically, the electric field is

$$\begin{aligned} \mathbf{E} \;&=\; -\nabla V \;-\; \frac{\partial \mathbf{A}}{\partial t} \;=\; \mathbf{0} \;-\; \frac{\mu_0 c N}{2\pi}\,\hat{\mathbf{z}}\,\frac{\partial}{\partial t}\frac{1}{\sqrt{(ct)^2 - s^2}} \\ &=\; -\frac{\mu_0 c N}{2\pi}\,\hat{\mathbf{z}}\,\frac{-c^2 t}{((ct)^2 - s^2)^{3/2}} \;=\; \frac{N}{2\pi\epsilon_0}\,\frac{(ct)}{((ct)^2 - s^2)^{3/2}}\,\hat{\mathbf{z}}\,, \end{aligned} \tag{S.88}$$

while the magnetic field is

$$\begin{aligned} \mathbf{B} \;&=\; \nabla \times \mathbf{A} \;=\; \frac{\mu_0 c N}{2\pi}\left(-\frac{\partial}{\partial s}\frac{1}{\sqrt{(ct)^2 - s^2}}\right)\hat{\boldsymbol{\phi}} \\ &=\; -\frac{\mu_0 c N}{2\pi}\,\frac{s}{((ct)^2 - s^2)^{3/2}}\,\hat{\boldsymbol{\phi}}. \end{aligned} \tag{S.89}$$