Problem **1**:

(a) At fixed observer time t,

$$\nabla_{\mathbf{r}}\cdot\mathbf{J}(\mathbf{r}',t_{\mathrm{ret}}) \;=\; \dot{\mathbf{J}}(\mathbf{r}',t_{\mathrm{ret}})\cdot\nabla_{\mathbf{r}}t_{\mathrm{ret}} \tag{S.1}$$

while

$$\nabla_{\mathbf{r}'}\cdot\mathbf{J}(\mathbf{r}',t_{\mathrm{ret}}) \;=\; \Big(\nabla\cdot\mathbf{J}\Big)_{@\text{fixed}\,t_{\mathrm{ret}}} \;+\; \dot{\mathbf{J}}(\mathbf{r}',t_{\mathrm{ret}})\cdot\nabla_{\mathbf{r}'}t_{\mathrm{ret}}\,. \tag{S.2}$$

hence

$$\nabla_{\mathbf{r}}\cdot\mathbf{J}(\mathbf{r}',t_{\mathrm{ret}}) \;+\; \nabla_{\mathbf{r}'}\cdot\mathbf{J}(\mathbf{r}',t_{\mathrm{ret}}) \;=\; \Big(\nabla\cdot\mathbf{J}\Big)_{@\text{fixed}\,t_{\mathrm{ret}}} \;+\; \dot{\mathbf{J}}(\mathbf{r}',t_{\mathrm{ret}})\cdot\big(\nabla_{\mathbf{r}}+\nabla_{\mathbf{r}'}\big)t_{\mathrm{ret}}\,. \tag{S.3}$$

But the retarded time $t_{\mathrm{ret}}=t-|\mathbf{r}-\mathbf{r}'|/c$ depends only on the difference between $\mathbf{r}$ and $\mathbf{r}'$, which means

$$\big(\nabla_{\mathbf{r}}+\nabla_{\mathbf{r}'}\big)t_{\mathrm{ret}} \;=\; 0, \tag{S.4}$$

so eq. (S.3) reduces to

$$\nabla_{\mathbf{r}}\cdot\mathbf{J}(\mathbf{r}',t_{\mathrm{ret}}) \;+\; \nabla_{\mathbf{r}'}\cdot\mathbf{J}(\mathbf{r}',t_{\mathrm{ret}}) \;=\; \Big(\nabla\cdot\mathbf{J}\Big)_{@\text{fixed}\,t_{\mathrm{ret}}} \;+\; 0. \tag{S.5}$$

Consequently,

$$\begin{aligned}
\nabla_{\mathbf{r}}\cdot\left(\frac{\mathbf{J}(\mathbf{r}',t_{\mathrm{ret}})}{\mathcal{R}}\right) \;+\; \nabla_{\mathbf{r}'}\cdot\left(\frac{\mathbf{J}(\mathbf{r}',t_{\mathrm{ret}})}{\mathcal{R}}\right) \;&=\; \frac{1}{\mathcal{R}}\nabla_{\mathbf{r}}\cdot\mathbf{J}(\mathbf{r}',t_{\mathrm{ret}}) \;-\; \frac{\nabla_{\mathbf{r}}\mathcal{R}}{\mathcal{R}^2}\cdot\cdot\mathbf{J}(\mathbf{r}',t_{\mathrm{ret}}) \;+\; \frac{1}{\mathcal{R}}\nabla_{\mathbf{r}'}\cdot\mathbf{J}(\mathbf{r}',t_{\mathrm{ret}}\\
&=\; \frac{1}{\mathcal{R}}\Big(\nabla_{\mathbf{r}}\cdot\mathbf{J}(\mathbf{r}',t_{\mathrm{ret}}) \;+\; \nabla_{\mathbf{r}'}\cdot\mathbf{J}(\mathbf{r}',t_{\mathrm{ret}})\Big) \;-\; \frac{1}{\mathcal{R}^2}\big(\nabla_{\mathbf{r}}+\nabla_{\mathbf{r}'}\big)\mathcal{R}\\
\text{where first term} \;&=\; \frac{1}{\mathcal{R}}\Big(\nabla\cdot\mathbf{J}\Big)_{@\text{fixed}\,t_{\mathrm{ret}}} \qquad \text{by eq. (S.5)}\\
\text{while second term} \;&=\; 0 \quad \text{because} \quad \big(\nabla_{\mathbf{r}}+\nabla_{\mathbf{r}'}\big)\big(\mathcal{R}=|\mathbf{r}-\mathbf{r}'|\big) \;=\; 0.
\end{aligned} \tag{S.6}$$

Altogether,

$$\nabla_{\mathbf{r}}\cdot\left(\frac{\mathbf{J}(\mathbf{r}',t_{\mathrm{ret}})}{\mathcal{R}}\right) \;+\; \nabla_{\mathbf{r}'}\cdot\left(\frac{\mathbf{J}(\mathbf{r}',t_{\mathrm{ret}})}{\mathcal{R}}\right) \;=\; \frac{1}{\mathcal{R}}\Big(\nabla\cdot\mathbf{J}\Big)_{@\text{fixed}\,t_{\mathrm{ret}}} \tag{S.7}$$

and hence eq. (2).

(b) The Landau gauge condition is

$$\nabla\cdot\mathbf{A}\;+\;\frac{1}{c^2}\frac{\partial V}{\partial t}\;=\;0. \tag{S.8}$$

For the potentials (1),

$$\begin{aligned}\frac{1}{c^2}\frac{\partial V}{\partial t}\;&=\;\frac{1}{4\pi\epsilon_0 c^2}\iiint d^3\mathrm{Vol}'\,\frac{\partial}{\partial t}\left(\frac{\rho(\mathbf{r}',t_{\mathrm{ret}})}{\mathcal{R}}\right), && \text{(S.9)}\\ \nabla\cdot\mathbf{A}\;&=\;\frac{\mu_0}{4\pi}\iiint d^3\mathrm{Vol}'\,\nabla_{\mathbf{r}}\cdot\left(\frac{\mathbf{J}(\mathbf{r}',t_{\mathrm{ret}})}{\mathcal{R}}\right), && \text{(S.10)}\end{aligned}$$

and since $1/(\epsilon_0 c^2)=\mu_0$,

$$\nabla\cdot\mathbf{A}\;+\;\frac{1}{c^2}\frac{\partial V}{\partial t}\;=\;\frac{\mu_0}{4\pi}\iiint d^3\mathrm{Vol}'\left[\frac{\partial}{\partial t}\left(\frac{\rho(\mathbf{r}',t_{\mathrm{ret}})}{\mathcal{R}}\right)\;+\;\nabla_{\mathbf{r}}\cdot\left(\frac{\mathbf{J}(\mathbf{r}',t_{\mathrm{ret}})}{\mathcal{R}}\right)\right]. \tag{S.11}$$

Now consider the expression inside the square brackets here. At fixed $\mathbf{r}$ and $\mathbf{r}'$, the $\mathcal{R}$ and the difference $t-t_{\mathrm{ret}}$ do not depend on t, hence

$$\frac{\partial}{\partial t}\left(\frac{\rho(\mathbf{r}',t_{\mathrm{ret}})}{\mathcal{R}}\right)\;=\;\frac{\dot\rho(\mathbf{r}',t_{\mathrm{ret}})}{\mathcal{R}}, \tag{S.12}$$

while

$$\nabla_{\mathbf{r}}\cdot\left(\frac{\mathbf{J}(\mathbf{r}',t_{\mathrm{ret}})}{\mathcal{R}}\right)\;=\;\nabla_{\mathbf{r}'}\cdot\left(\frac{\mathbf{J}(\mathbf{r}',t_{\mathrm{ret}})}{\mathcal{R}}\right)\;+\;\frac{(\nabla\cdot\mathbf{J})(\mathbf{r}',t_{\mathrm{ret}})}{\mathcal{R}} \tag{2}$$

Altogether,

$$\Big[\cdots\Big]\;=\;\frac{1}{\mathcal{R}}(\dot\rho+\nabla\cdot\mathbf{J})@(\mathbf{r}',t_{\mathrm{ret}})\;+\;\nabla_{\mathbf{r}'}\cdot\left(\frac{\mathbf{J}(\mathbf{r}',t_{\mathrm{ret}})}{\mathcal{R}}\right), \tag{S.13}$$

where the first term on the RHS vanishes by the continuity equation $\dot\rho+\nabla\cdot\mathbf{J}=0$. Thus,

$$\Big[\cdots\Big]\;=\;\nabla_{\mathbf{r}'}\cdot\left(\frac{\mathbf{J}(\mathbf{r}',t_{\mathrm{ret}})}{\mathcal{R}}\right)\;+\;0, \tag{S.14}$$

so eq. (S.11) reduces to

$$\begin{aligned}\nabla\cdot\mathbf{A}\;+\;\frac{1}{c^2}\frac{\partial V}{\partial t}\;&=\;\frac{\mu_0}{4\pi}\iiint d^3\mathrm{Vol}'\,\nabla_{\mathbf{r}'}\cdot\left(\frac{\mathbf{J}(\mathbf{r}',t_{\mathrm{ret}})}{\mathcal{R}}\right)\\ &\qquad\langle\!\langle\text{ by Gauss Theorem }\rangle\!\rangle\\ &=\;\frac{\mu_0}{4\pi}\oiint\frac{\mathbf{J}(\mathbf{r}',t_{\mathrm{ret}})}{\mathcal{R}}\cdot\mathbf{d^2a}',\end{aligned} \tag{S.15}$$

where the last integral is over the boundary surface approaching infinity. (For example, the

$R \to \infty$ limit of the integral over a sphere of very large radius R.) Assuming all currents are limited to a finite volume (however large it might be), the integral here vanishes for a large enough boundary surface, thus the potentials indeed obey the Landau gauge condition (S.8). *Quod erat demonstrandum.*

Problem **2**:

As the ring rotates, the potential (3) rotates with it, thus

$$\lambda(\phi,t) \;=\; \lambda_0 \left|\sin\frac{\phi-\omega t}{2}\right|, \tag{S.16}$$

or in 3D terms (in cylindrical coordinates)

$$\rho(s,\phi,z,t) \;=\; \lambda_0 \left|\sin\frac{\phi-\omega t}{2}\right| \delta(z)\delta(s-a). \tag{S.17}$$

Since this charge density is time dependent, we also have a current in the $\hat{\boldsymbol{\phi}}$ direction. It specific form, follows from the continuity equation:

$$\mathbf{J}(s,\phi,z,t) \;=\; a\omega\lambda_0 \left|\sin\frac{\phi-\omega t}{2}\right| \delta(z)\delta(s-a)\hat{\boldsymbol{\phi}}, \tag{S.18}$$

so that

$$\dot{\rho} \;+\; \nabla\cdot\mathbf{J} \;=\; \dot{\rho} \;+\; \frac{1}{s}\frac{\partial J_\phi}{\partial\phi} \;=\; 0. \tag{S.19}$$

Now let's plug these charges and current into eqs. (1) for the scalar and vector potentials. For the observer $\mathbf{r}$ at the center of the ring, the distance $\mathcal{R}$ to all points of the ring is fixed at $\mathcal{R} \equiv a$. Consequently, the retarded time is at fixed delay $t - t_{\text{ret}} = a/c$ behind the observer time, so for any give observer time t, we have a constant retarded time $t_{\text{ret}} = t - (a/c)$ inside the integral. Thus,

$$\rho(s',\phi',z',t_{\rm ret}) \;=\; \lambda_0 \left|\sin\frac{\phi'-\omega t+\omega a/c}{2}\right| \delta(z)\delta(s-a) \tag{S.20}$$

and

$$\mathbf{J}(s',\phi',z',t_{\rm ret}) \;=\; a\omega\lambda_0 \left|\sin\frac{\phi'-\omega t+\omega a/c}{2}\right| \delta(z)\delta(s-a)\hat{\boldsymbol{\phi}}\,. \tag{S.21}$$

Consequently, the scalar potential (1) at the center $\mathbf{0}$ evaluates to

$$\begin{aligned} V(\mathbf{0},t) \;&=\; \frac{1}{4\pi\epsilon_0}\iiint d^3{\rm Vol}'\,\frac{\lambda_0}{a}\left|\sin\frac{\phi'-\omega t+\omega a/c}{2}\right|\delta(z)\delta(s-a) \\ &=\; \frac{1}{4\pi\epsilon_0}\oint a\,d\phi'\,\frac{\lambda_0}{a}\left|\sin\frac{\phi'-\omega t+\omega a/c}{2}\right| \\ &=\; \frac{\lambda_0}{4\pi\epsilon_0}\oint d\phi'\,\left|\sin\frac{\phi'-\omega t+\omega a/c}{2}\right| \end{aligned} \tag{S.22}$$

where the integral is over any 2π period of ϕ, starting with any ϕ_0 we like and ending at $\phi_0+2\pi$. Letting $\phi_0=\omega(t-a/c)$ and shifting the integration variable to $\alpha=\phi-\phi_0$, we have

$$\begin{aligned} \oint\cdots \;&=\; \int\limits_0^{2\pi} d\alpha\,\left|\sin\frac{\alpha}{2}\right| \\ &\qquad\langle\!\langle\text{ since } \sin(\alpha/2)\geq 0 \text{ for } 0\leq\alpha\leq 2\pi\,\rangle\!\rangle \\ &=\; \int\limits_0^{2\pi} d\alpha\,\sin(\alpha/2) \;=\; 2\int\limits_0^{\pi} d(\alpha/2)\sin(\alpha/2) \\ &=\; 2\Big(-\cos(\pi)+\cos(0)\Big) \;=\; 4, \end{aligned} \tag{S.23}$$

and therefore

$$V(\mathbf{0},t) \;=\; \frac{\lambda_0}{\pi\epsilon_0}\,. \tag{S.24}$$

Likewise, for the vector potential at the ring's center we have

$$\begin{aligned}
\mathbf{A}(\mathbf{0},t) &= \frac{\mu_0}{4\pi}\iiint d^3\mathrm{Vol}'\,\frac{a\omega\lambda_0}{a}\left|\sin\frac{\phi'-\omega t+\omega a/c}{2}\right|\delta(z)\delta(s-a)\hat{\boldsymbol{\phi}}\\
&= \frac{\mu_0}{4\pi}\oint a\,d\phi'\,\frac{a\omega\lambda_0}{a}\left|\sin\frac{\phi'-\omega t+\omega a/c}{2}\right|\hat{\boldsymbol{\phi}}\\
&= \frac{\mu_0\omega a\lambda_0}{4\pi}\oint d\phi'\left|\sin\frac{\phi'-\omega t+\omega a/c}{2}\right|\hat{\boldsymbol{\phi}}.
\end{aligned} \tag{S.25}$$

This integral is a bit more complicated than (S.23) because the unit vector $\hat{\boldsymbol{\phi}}$ depends on the angle ϕ:

$$\hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}}. \tag{S.26}$$

Consequently, identifying $\phi = \alpha + \phi_0$ for $\phi_0 = \omega(t - a/c)$, we have

$$\oint\cdots = \int\limits_0^{2\pi} d\alpha, \sin(\alpha/0)\Big(-\sin(\alpha+\phi_0)\hat{\mathbf{x}} + \cos(\alpha+\phi_0)\hat{\mathbf{y}}\Big) \tag{S.27}$$

where

$$\begin{aligned}
\int\limits_0^{2\pi} d\alpha\,\sin(\alpha/0)\sin(\alpha+\phi_0) &= \int\limits_0^{2\pi} d\alpha\Big(\tfrac{1}{2}\cos(\tfrac{1}{2}\alpha+\phi_0) - \tfrac{1}{2}\cos(\tfrac{3}{2}\alpha+\phi_0)\Big)\\
&= \Big(\sin(\tfrac{1}{2}\alpha+\phi_0) - \tfrac{1}{3}\sin(\tfrac{3}{2}\alpha+\phi_0)\Big)\Big|_{\alpha=0}^{\alpha=2\pi}\\
&= \Big(\sin(\pi+\phi_0) - \tfrac{1}{3}\sin(3\pi+\phi_0)\Big) - \Big(\sin(\phi_0) - \tfrac{1}{3}\sin(\phi_0)\Big)\\
&= -\tfrac{4}{3}\sin(\phi_0),
\end{aligned} \tag{S.28}$$

while

$$\begin{aligned}
\int\limits_0^{2\pi} d\alpha\,\sin(\alpha/0)\cos(\alpha+\phi_0) &= \int\limits_0^{2\pi} d\alpha\Big(-\tfrac{1}{2}\sin(\tfrac{1}{2}\alpha+\phi_0) + \tfrac{1}{2}\sin(\tfrac{3}{2}\alpha+\phi_0)\Big)\\
&= \Big(\cos(\tfrac{1}{2}\alpha+\phi_0) - \tfrac{1}{3}\cos(\tfrac{3}{2}\alpha+\phi_0)\Big)\Big|_{\alpha=0}^{\alpha=2\pi}\\
&= \Big(\cos(\pi+\phi_0) - \tfrac{1}{3}\cos(3\pi+\phi_0)\Big) - \Big(\cos(\phi_0) - \tfrac{1}{3}\cos(\phi_0)\Big)\\
&= -\tfrac{4}{3}\cos(\phi_0),
\end{aligned} \tag{S.29}$$

thus

$$\oint\cdots\ =\ +\tfrac{4}{3}\sin(\phi_0)\hat{\mathbf{x}}\ -\ \tfrac{4}{3}\cos(\phi_0)\hat{\mathbf{y}} \tag{S.30}$$

and therefore

$$\mathbf{A}(\mathbf{0},t)\ =\ \frac{\mu_0\omega a\lambda_0}{3\pi}\Big(\sin(\omega(t-a/c))\hat{\mathbf{x}}\ -\ \cos(\omega(t-a/c))\hat{\mathbf{y}}\Big). \tag{S.31}$$

<u>Problem **3**</u>:

Jefimenko equation for the magnetic field of a time-dependent current (*cf.* my notes on EM potentials, eq. (65) on page 13, or textbook §10.2, eq. (10.38) on page 450) says

$$\mathbf{B}(\mathbf{r},t)\ =\ \frac{\mu_0}{4\pi}\iiint d^3\mathrm{Vol}'\left(\frac{\mathbf{n}}{\mathcal{R}^2}\times\mathbf{J}(\mathbf{r}',t_{\mathrm{ret}})\ +\ \frac{\mathbf{n}}{c\mathcal{R}}\times\dot{\mathbf{J}}(\mathbf{r}',t_{\mathrm{ret}})\right). \tag{S.32}$$

For a slowly changing current $\mathbf{J}(\mathbf{r},t)$, we may approximate

$$\begin{aligned}\mathbf{J}(\mathbf{r}',t_{\mathrm{ret}})\ &=\ \mathbf{J}(\mathbf{r}',t)\ -\ (t-t_{\mathrm{ret}})\dot{\mathbf{J}}(\mathbf{r}',t)\ +\ \tfrac{1}{2}(t-t_{\mathrm{ret}})^2\ddot{\mathbf{J}}(\mathbf{r}',t)\ +\ \cdots\\ &=\ \mathbf{J}(\mathbf{r}',t)\ -\ \frac{\mathcal{R}}{c}\dot{\mathbf{J}}(\mathbf{r}',t)\ +\ \frac{\mathcal{R}^2}{2c^2}\ddot{\mathbf{J}}(\mathbf{r}',t)\ +\ \cdots,\end{aligned} \tag{S.33}$$

and likewise

$$\dot{\mathbf{J}}(\mathbf{r}',t_{\mathrm{ret}})\ =\ \dot{\mathbf{J}}(\mathbf{r}',t)\ -\ \frac{\mathcal{R}}{c}\ddot{\mathbf{J}}(\mathbf{r}',t)\ +\ \cdots. \tag{S.34}$$

Consequently,

$$\mathbf{J}(\mathbf{r}',t_{\mathrm{ret}})\ +\ \frac{\mathcal{R}}{c}\dot{\mathbf{J}}(\mathbf{r}',t_{\mathrm{ret}})\ =\ \mathbf{J}(\mathbf{r}',t)\ -\ \frac{\mathcal{R}^2}{2c^2}\ddot{\mathbf{J}}(\mathbf{r}',t)\ +\ \cdots\ =\ \mathbf{J}(\mathbf{r}',t)\ +\ O\left(\frac{\mathbf{J}\mathcal{R}^2}{c^2\tau^2}\right) \tag{S.35}$$

where τ is the time scale at which the current changes, thus $\dot{\mathbf{J}}\sim\mathbf{J}/\tau$, $\ddot{\mathbf{J}}\sim\mathbf{J}/\tau^2$, *etc.* Note that as a power series in $(\mathcal{R}/c\tau)$, the series (S.35) has a zeroth-order term and then the second-order and the higher-order terms, but the first-order term cancels out between the $\mathbf{J}(t_{\mathrm{ret}})$ and the $(\mathcal{R}/c)\dot{\mathbf{J}}(t_{\mathrm{ret}})$.

In the context of the Jefimenko equation (S.32), this cancellation means

$$\begin{aligned}\mathbf{B}(\mathbf{r},t) &= \frac{\mu_0}{4\pi}\iiint d^3\mathrm{Vol}'\,\frac{\mathbf{n}}{\mathcal{R}^2}\times\left(\mathbf{J}(\mathbf{r}',t_{\mathrm{ret}})\,+\,\frac{\mathcal{R}}{c}\,\dot{\mathbf{J}}(\mathbf{r}',t_{\mathrm{ret}})\right)\\ &= \frac{\mu_0}{4\pi}\iiint d^3\mathrm{Vol}'\,\frac{\mathbf{n}}{\mathcal{R}^2}\times\left(\mathbf{J}(\mathbf{r}',t)\,+\,O\left(\frac{\mathbf{J}\mathcal{R}^2}{c^2\tau^2}\right)\right),\end{aligned}\tag{S.36}$$

where the leading term inside the $(\cdots)$ is the current $\mathbf{J}(t)$ at the observer time, while the first-order in $(\mathcal{R}/c\tau)$ correction cancels out. Thus, the difference between the exact Jefimenko equation and the quasi-static Biot–Savart–Laplace approximation

$$\mathbf{B}(\mathbf{r},t) \;=\; \frac{\mu_0}{4\pi}\iiint d^3\mathrm{Vol}'\,\frac{\mathbf{n}}{\mathcal{R}^2}\times\mathbf{J}(\mathbf{r}',t)\tag{S.37}$$

is of the second order in $(\mathcal{R}/c\tau)$ rather than of the first order.

<u>Problem **4**</u>:

Let's start with the scalar potential:

$$\begin{aligned}V(\mathbf{r},t) &= \frac{1}{4\pi\epsilon_0}\iiint d^3\mathrm{Vol}'\,\frac{-\mathbf{p}(t_{\mathrm{ret}})}{\mathcal{R}}\cdot\nabla\delta^{(3)}(\mathbf{r}')\\ &\qquad\langle\!\langle\,\text{integrating by parts}\,\rangle\!\rangle\\ &= \frac{1}{4\pi\epsilon_0}\iiint d^3\mathrm{Vol}'\,\left(\nabla_{\mathbf{r}'}\cdot\frac{\mathbf{p}(t_{\mathrm{ret}})}{\mathcal{R}}\right)\delta^{(3)}(\mathbf{r}')\\ &= \frac{1}{4\pi\epsilon_0}\left[\nabla_{\mathbf{r}'}\cdot\frac{\mathbf{p}(t_{\mathrm{ret}})}{\mathcal{R}}\right]_{@\mathbf{r}'=0}\end{aligned}\tag{S.38}$$

where

$$\nabla_{\mathbf{r}'}\cdot\frac{\mathbf{p}(t_{\mathrm{ret}})}{\mathcal{R}} \;=\; -\frac{(\nabla_{\mathbf{r}'}\mathcal{R})}{\mathcal{R}^2}\cdot\mathbf{p}(t_{\mathrm{ret}})\;+\;\frac{(\nabla_{\mathbf{r}'}t_{\mathrm{ret}})}{\mathcal{R}}\,\frac{\partial\mathbf{p}}{\partial t_{\mathrm{ret}}}\;=\;+\frac{\mathbf{n}}{\mathcal{R}^2}\cdot\mathbf{p}(t_{\mathrm{ret}})\;+\;\frac{\mathbf{n}}{c\mathcal{R}}\cdot\dot{\mathbf{p}}(t_{\mathrm{ret}}).\tag{S.39}$$

Having taken this derivative, we now plug $\mathbf{r}'=0$, hence $\mathcal{R}=|\mathbf{r}|=r$ while $\mathbf{n}=\mathbf{r}/r$.

Consequently, the scalar potential is

$$V(\mathbf{r},t) \;=\; \frac{1}{4\pi\epsilon_0}\left[\frac{\mathbf{n}\cdot\mathbf{p}}{r^2} + \frac{\mathbf{n}\cdot\dot{\mathbf{p}}}{cr}\right]_{\rm ret} . \tag{8}$$

Next, the vector potential:

$$\begin{aligned}
\mathbf{A}(\mathbf{r},t) \;&=\; \frac{\mu_0}{4\pi}\iiint d^3\mathrm{Vol}'\,\frac{\dot{\mathbf{p}}(t_{\rm ret})}{\mathcal{R}}\,\delta^{(3)}(\mathbf{r}') \\
&=\; \frac{\mu_0}{4\pi}\left[\frac{\dot{\mathbf{p}}(t_{\rm ret})}{\mathcal{R}}\right]_{@\mathbf{r}'=0} \\
&=\; \frac{\mu_0}{4\pi}\left[\frac{\dot{\mathbf{p}}}{r}\right]_{\rm ret} .
\end{aligned} \tag{9}$$

Now, the magnetic field:

$$\begin{aligned}
\mathbf{B}(\mathbf{r},t) \;&=\; \nabla\times\mathbf{A}(\mathbf{r},t) \;=\; \frac{\mu_0}{4\pi}\,\nabla\times\frac{\dot{\mathbf{p}}(t_{\rm ret})}{r} \\
&=\; \frac{\mu_0}{4\pi}\left(-\frac{\nabla r}{r^2}\times\dot{\mathbf{p}}(t_{\rm ret}) \;+\; \frac{\nabla t_{\rm ret}}{r}\times\frac{\partial\dot{\mathbf{p}}}{\partial t_{\rm ret}}\right) \\
&=\; \frac{\mu_0}{4\pi}\left[-\frac{\mathbf{n}}{r^2}\times\dot{\mathbf{p}} \;-\; \frac{\mathbf{n}}{cr}\times\ddot{\mathbf{p}}\right]_{\rm ret} .
\end{aligned} \tag{11}$$

Finally, the electric field

$$\mathbf{E} \;=\; -\nabla V \;-\; \frac{\partial\mathbf{A}}{\partial t} \tag{S.40}$$

where

$$\frac{\partial\mathbf{A}}{\partial t} \;=\; \frac{\mu_0}{4\pi}\left(\frac{\partial}{\partial t}\frac{\dot{\mathbf{p}}(t_{\rm ret})}{r}\right)_{@\text{fixed}\,\mathbf{r}} \;=\; \frac{1}{4\pi\epsilon_0 c^2}\,\frac{\ddot{\mathbf{p}}(t_{\rm ret})}{r}\,, \tag{S.41}$$

while

$$\nabla V \;=\; \frac{1}{4\pi\epsilon_0}\,\nabla\left[\frac{\mathbf{n}\cdot\mathbf{p}(t_{\rm ret})}{r^2} + \frac{\mathbf{n}\cdot\dot{\mathbf{p}}(t_{\rm ret})}{cr}\right]. \tag{S.42}$$

Note that the gradient here is taken at fixed observer time t rather than fixed retarded time

t_{ret}, thus

$$\nabla_i p_j(t_{\mathrm{ret}}) \;=\; (\nabla_i t_{\mathrm{ret}})\,\frac{\partial p_j}{\partial t_{\mathrm{ret}}} \;=\; -\frac{n_i}{c}\,\dot{p}_j(t_{\mathrm{ret}}) \tag{S.43}$$

and likewise

$$\nabla_i \dot{p}_j(t_{\mathrm{ret}}) \;=\; -\frac{n_i}{c}\,\ddot{p}_j(t_{\mathrm{ret}}). \tag{S.44}$$

Consequently,

$$\begin{aligned}
\nabla_i\left(\frac{\mathbf{n}\cdot\mathbf{p}(t_{\mathrm{ret}})}{r^2}\right) &= \left(\nabla_i\frac{n_j}{r^2}\right)p_j(t_{\mathrm{ret}}) \;+\; \frac{n_j}{r^2}\,\nabla_i p_j(t_{\mathrm{ret}}) \\
&= \frac{\delta_{ij}-3n_in_j}{r^3}\,p_j(t_{\mathrm{ret}}) \;-\; \frac{n_j}{r^2}\,\frac{n_i}{c}\,\dot{p}_j(t_{\mathrm{ret}}) \\
&= \left[\frac{\mathbf{p}(t_{\mathrm{ret}}) \,-\, 3\mathbf{n}(\mathbf{n}\cdot\mathbf{p}(t_{\mathrm{ret}}))}{r^3} \,-\, \frac{\mathbf{n}(\mathbf{n}\cdot\dot{\mathbf{p}}(t_{\mathrm{ret}}))}{cr^2}\right]_i .
\end{aligned} \tag{S.45}$$

while

$$\begin{aligned}
\nabla_i\left(\frac{\mathbf{n}\cdot\dot{\mathbf{p}}(t_{\mathrm{ret}})}{cr}\right) &= \left(\nabla_i\frac{n_j}{cr}\right)\dot{p}_j(t_{\mathrm{ret}}) \;+\; \frac{n_j}{cr}\,\nabla_i \dot{p}_j(t_{\mathrm{ret}}) \\
&= \frac{\delta_{ij}-2n_in_j}{cr^2}\,p_j(t_{\mathrm{ret}}) \;-\; \frac{n_j}{cr}\,\frac{n_i}{c}\,\ddot{p}_j(t_{\mathrm{ret}}) \\
&= \left[\frac{\dot{\mathbf{p}}(t_{\mathrm{ret}}) \,-\, 2\mathbf{n}(\mathbf{n}\cdot\dot{\mathbf{p}}(t_{\mathrm{ret}}))}{cr^2} \,-\, \frac{\mathbf{n}(\mathbf{n}\cdot\ddot{\mathbf{p}}(t_{\mathrm{ret}}))}{c^2r}\right]_i .
\end{aligned} \tag{S.46}$$

Altogether, this gives us

$$\nabla V(\mathbf{r},t) \;=\; \frac{1}{4\pi\epsilon_0}\left[\frac{\mathbf{p} \,-\, 3\mathbf{n}(\mathbf{n}\cdot\mathbf{p})}{r^3} \,+\, \frac{\dot{\mathbf{p}} \,-\, 3\mathbf{n}(\mathbf{n}\cdot\dot{\mathbf{p}})}{cr^2} \,-\, \frac{\mathbf{n}(\mathbf{n}\cdot\ddot{\mathbf{p}})}{c^2r}\right]_{\mathrm{ret}} \tag{S.47}$$

and consequently

$$\begin{aligned}
\mathbf{E}(\mathbf{r},t) &= -\nabla V(\mathbf{r},t) \;-\; \frac{\partial\mathbf{A}(\mathbf{r},t)}{\partial t} \\
&= -\frac{1}{4\pi\epsilon_0}\left[\frac{\mathbf{p} \,-\, 3\mathbf{n}(\mathbf{n}\cdot\mathbf{p})}{r^3} \,+\, \frac{\dot{\mathbf{p}} \,-\, 3\mathbf{n}(\mathbf{n}\cdot\dot{\mathbf{p}})}{cr^2} \,+\, \frac{\ddot{\mathbf{p}} \,-\, \mathbf{n}(\mathbf{n}\cdot\ddot{\mathbf{p}})}{c^2r}\right]_{\mathrm{ret}} .
\end{aligned} \tag{10}$$

Problem **5**:

The observer at the ball's center and time t, sees the point $\mathbf{r}'$ of the charged ball back when it was at the retarded time $t_{\text{ret}} = t - |\mathbf{r}'|/c$, so the ball's charge density was

$$\rho(t_{\text{ret}}) \;=\; \frac{3Q}{4\pi R^3(t_{\text{ret}})} \;=\; \frac{(3/4\pi)Q}{v^3[t-(r'/c)]^3}\,. \tag{S.48}$$

He also sees the ball as having radius

$$a \;=\; vt_{\text{ret}} \;=\; v(t-a/c) \quad\Longrightarrow\quad a(t) \;=\; \frac{vt}{1+(v/c)}. \tag{S.49}$$

Consequently,

$$\begin{aligned}
Q_{\text{eff}} &= \iiint\limits_{\substack{\text{visible}\\ \text{ball}}} d^3\text{Vol}'\,\rho(t_{\text{ret}}(\mathbf{r}')) \\
&= \int\limits_0^a 4\pi r'^2 dr' * \frac{(3/4\pi)Q}{v^3[t-(r'/c)]^3} \\
&\qquad \langle\!\langle \text{ changing integration variable from } r' \text{ to } x = ct - r' \,\rangle\!\rangle \\
&= \frac{3Q}{v^3}\int\limits_{ct-a}^{ct} dx\,(ct-x)^2\,\frac{c^3}{x^3} \\
&= \frac{3Qc^3}{v^3}\left[-\frac{(ct)^2}{2x^2} + \frac{2ct}{x} + \ln(x)\right]_{x=(ct-a)}^{a=ct} \\
&= \frac{3Qc^3}{v^3}\left[\frac{1}{2}\left(\frac{(ct)^2}{(ct-a)^2}-1\right) - 2\left(\frac{ct}{ct-a}-1)\right) + \ln\frac{ct}{ct-a}\right]
\end{aligned} \tag{S.50}$$

In this formula,

$$ct \;-\; a \;=\; ct \;-\; \frac{vt}{1+(v/c)} \;=\; \frac{ct}{1+(v/c)} \quad\Longrightarrow\quad \frac{ct}{ct-a} \;=\; 1 \;+\; (v/c), \tag{S.51}$$

hence the expression inside $[\cdots]$ on the bottom line of eq. (S.50) evaluates to

$$\Big[\cdots\Big] \;=\; \tfrac{1}{2}(v/c)^2 \;+\; (v/c) \;-\; 2(v/c) \;+\; \ln(1+(v/c)), \tag{S.52}$$

therefore

$$\frac{Q_{\mathrm{eff}}}{Q} \;=\; \frac{3}{(v/c)^3}\Big[\ln(1+(v/c)) \;-\; (v/c) \;+\; \tfrac{1}{2}(v/c)^2\Big]. \tag{S.53}$$

Eq. (S.53) gives the general formula for any expansion speed, slow or fast. For slow expansion (compared to the speed of light), we approximate

$$\ln((1+(v/c)) \;\approx\; (v/c) \;-\; \tfrac{1}{2}(v/c)^2 \;+\; \tfrac{1}{3}(v/c)^3 \;-\; \tfrac{1}{4}(v/c)^4 \;+\; \cdots, \tag{S.54}$$

hence

$$\ln(1+(v/c)) \;-\; (v/c) \;+\; \tfrac{1}{2}(v/c)^2 \;=\; \tfrac{1}{3}(v/c)^3 \;-\; \tfrac{1}{4}(v/c)^4 \;+\; \cdots, \tag{S.55}$$

and therefore

$$\frac{Q_{\mathrm{eff}}}{Q} \;=\; 1-\; \tfrac{3}{4}(v/c) \;+\; \cdots. \tag{S.56}$$

On the other hand, for the expansion almost at the speed of light, $v \approx c$, we have

$$\ln(1+(v/c)) \;-\; (v/c) \;+\; \tfrac{1}{2}(v/c)^2 \;\approx\; \ln(2) \;-\; 1 \;+\; \tfrac{1}{2} \;\approx\; 0.193, \tag{S.57}$$

hence

$$\frac{Q_{\mathrm{eff}}}{Q} \;\approx\; 3\big(\ln(2) \;-\; \tfrac{1}{2}\big) \;\approx\; 0.58. \tag{S.58}$$

<u>Problem **6**</u>:

(a) For the tail end of the rod, the retarded time obtains from

$$ct \;-\; ct_{\mathrm{ret}} \;=\; x_1(t_{\mathrm{ret}}) \;=\; vt_{\mathrm{ret}}, \tag{S.59}$$

thus

$$t_{\mathrm{ret}}^{\mathrm{tail}} \;=\; \frac{ct}{c+v} \quad\Longrightarrow\quad x_{\mathrm{ret}}^{\mathrm{tail}} \;\overset{\mathrm{def}}{=}\; x_1(t_{\mathrm{ret}}^{\mathrm{tail}}) \;=\; \frac{vct}{c+v}\,. \tag{S.60}$$

Likewise, for the head of the rod, we have

$$ct\ -\ ct_{\text{ret}}\ =\ x_2(t_{\text{ret}})\ =\ vt_{\text{ret}}\ +\ L, \tag{S.61}$$

hence

$$t_{\text{ret}}^{\text{head}}\ =\ \frac{ct-L}{c+v}\ \implies\ x_{\text{ret}}^{\text{head}}\ \stackrel{\text{def}}{=}\ x_2(t_{\text{ret}}^{\text{head}})\ =\ v\times\frac{ct-L}{c+v}\ +\ L\ =\ \frac{c(vt+L)}{c+v}. \tag{S.62}$$

(b) At the coordinate origin $\mathbf{r}=\mathbf{0}$, the scalar potential is

$$V(\mathbf{0},t)\ =\ \frac{1}{4\pi\epsilon_0}\iiint d^3\text{Vol}'\,\frac{\rho(\mathbf{r}',t_{\text{ret}})}{|\mathbf{r}'|}\ =\ \frac{1}{4\pi\epsilon_0}\int dx'\,\frac{\lambda(x',t_{\text{ret}})}{|x'|} \tag{S.63}$$

where

$$\lambda(x',t_{\text{ret}})\ =\ \begin{cases}(Q/L) & \text{when } x_1(t_{\text{ret}})<x'<x_2(t_{\text{ret}}),\\ 0 & \text{otherwise.}\end{cases} \tag{S.64}$$

Or in other words,

$$\lambda(x',t_{\text{ret}})\ =\ \begin{cases}(Q/L) & \text{for } x_{\text{ret}}^{\text{tail}}<x'<x_{\text{ret}}^{\text{head}},\\ 0 & \text{otherwise,}\end{cases} \tag{S.65}$$

hence

$$V(\mathbf{0},t)\ =\ \frac{Q/L}{4\pi\epsilon_0}\int\limits_{x_{\text{ret}}^{\text{tail}}}^{x_{\text{ret}}^{\text{head}}}\frac{dx'}{x'}\ =\ \frac{Q/L}{4\pi\epsilon_0}\ln\frac{x_{\text{ret}}^{\text{head}}}{x_{\text{ret}}^{\text{tail}}}\ =\ \frac{Q/L}{4\pi\epsilon_0}\ln\frac{vt+L}{vt}. \tag{S.66}$$

Now consider the vector potential at the coordinate origin,

$$\mathbf{A}(\mathbf{0},t)\ =\ \frac{1}{4\pi\epsilon_0c^2}\iiint d^3\text{Vol}'\,\frac{\mathbf{J}(\mathbf{r}',t_{\text{ret}})}{|\mathbf{r}'|}. \tag{S.67}$$

Since the rod is moving at uniform velocity $\mathbf{v}=\mathbf{v}\hat{\mathbf{x}}$, we have

$$\mathbf{J}(\mathbf{r}',t_{\text{ret}})\ =\ \mathbf{v}\rho(\mathbf{r}',t_{\text{ret}}), \tag{S.68}$$

and therefore

$$\mathbf{A}(\mathbf{0},t) \;=\; \frac{\mathbf{v}}{4\pi\epsilon_0 c^2}\iiint d^3\mathrm{Vol}'\,\frac{\rho(\mathbf{r}',t_{\mathrm{ret}})}{|\mathbf{r}'|} \;=\; \frac{\mathbf{v}}{c^2}\,V(\mathbf{0},t). \tag{S.69}$$

Thus, in light of eq. (S.66),

$$\mathbf{A}(\mathbf{0},t) \;=\; \frac{(Q/L)v\hat{\mathbf{x}}}{4\pi\epsilon_0}\,\ln\frac{vt+L}{vt}\,. \tag{S.70}$$

(c) In the short rod limit (compared to the distance to the rod), $L \ll vt$¡ we have

$$\ln\frac{vt+L}{vt} \;\approx\; \frac{L}{vt}\,, \tag{S.71}$$

hence

$$\begin{aligned} V(\mathbf{0},t) &\approx \frac{Q}{4\pi\epsilon_0}\,\frac{1}{vt}\,,\\ \mathbf{A}(\mathbf{0},t) &\approx \frac{Qv\hat{\mathbf{x}}}{4\mu_0}\,\frac{1}{vt}\,. \end{aligned} \tag{S.72}$$

By comparison, the Liénard–Wiechert potentials for a point charge Q moving at the same uniform velocity are

$$\begin{aligned} V(\mathbf{0},t) &\approx \frac{Q}{4\pi\epsilon_0}\,\frac{1}{\mathcal{R}(1+v/c)}\,,\\ \mathbf{A}(\mathbf{0},t) &\approx \frac{Qv\hat{\mathbf{x}}}{4\mu_0}\,\frac{1}{\mathcal{R}(1+v/c)}\,, \end{aligned} \tag{S.73}$$

where $(1+v/c)$ factor is the $(1-\mathbf{n}\cdot\mathbf{v}/c)$ for the charge moving directly away from the observer. Also, in the Liénard–Wiechert potentials (S.73), the distance $\mathcal{R}$ is evaluated at the retarded time, thus for the observer at the origin,

$$\mathcal{R} \;=\; vt_{\mathrm{ret}} \;=\; \frac{vt}{1+(v/c)}\,. \tag{S.74}$$

Consequently, the two non-quasi-static factors of the Liénard–Wiechert potentials — the $1/(1-\mathbf{n}\cdot\mathbf{v}/c)$ factor, and the distance calculated at the retarded time — happen to precisely cancel each other since

$$\mathcal{R}\times(1+(v/c)) \;=\; vt \;=\; r. \tag{S.75}$$

Thus, for the case at hand, the Liénard–Wiechert potentials (S.73)become simply

$$\begin{aligned} V(\mathbf{0},t) &\approx \frac{Q}{4\pi\epsilon_0}\,\frac{1}{vt}\,,\\ \mathbf{A}(\mathbf{0},t) &\approx \frac{Qv\hat{\mathbf{x}}}{4\mu_0}\,\frac{1}{vt}\,, \end{aligned} \tag{S.76}$$

in perfect agreement with the $L\to 0$ limit (S.72) of the potentials we got in part (b).

<u>Problem</u> **7**:

The Liénard–Wiechert potentials for a point charge Q moving at constant velocity $\mathbf{v}$ we work out in class in some detail, *cf.* my notes on EM potentials, eqs. (110) on page 21:

$$V(\mathbf{r},t) \;=\; \frac{Q}{4\pi\epsilon_0}\,\frac{c}{\sqrt{\mathcal{D}(\mathbf{r},t)}}, \qquad \mathbf{A}(\mathbf{r},t) \;=\; \frac{Q}{4\pi\epsilon_0 c}\,\frac{\mathbf{v}}{\sqrt{\mathcal{D}(\mathbf{r},t)}} \;=\; \frac{\mathbf{v}}{c^2}V(\mathbf{r},t) \tag{S.77}$$

where

$$\mathcal{D}(\mathbf{r},t) \;\stackrel{\text{def}}{=}\; (c^2t-\mathbf{v}\cdot\mathbf{r})^2 \;+\; (c^2-v^2)(r^2-c^2t^2)\,. \tag{S.78}$$

To verify the Landau gauge condition

$$\nabla\cdot\mathbf{A} \;+\; \frac{1}{c^2}\,\frac{\partial V}{\partial t} \;=\; 0, \tag{S.79}$$

we evaluate

$$\nabla\cdot\mathbf{A} \;=\; \frac{Q}{4\pi\epsilon_0 c}\,\frac{-1/2}{\mathcal{D}^{3/2}}\,(\mathbf{v}\cdot\nabla)\mathcal{D}, \tag{S.80}$$

$$\frac{1}{c^2}\,\frac{\partial V}{\partial t} \;=\; \frac{Q}{4\pi\epsilon_0 c}\,\frac{-1/2}{\mathcal{D}^{3/2}}\,\frac{\partial\mathcal{D}}{\partial t}\,, \tag{S.81}$$

so the Landau gauge condition (S.79) amounts to

$$0 \;=\; \nabla\cdot\mathbf{A} \;+\; \frac{1}{c^2}\frac{\partial V}{\partial t} \;=\; \frac{Q}{4\pi\epsilon_0 c}\,\frac{-1/2}{\mathcal{D}^{3/2}}\left((\mathbf{v}\cdot\nabla\;+\;\frac{\partial}{\partial t}\right)\mathcal{D}(\mathbf{r},t) \tag{S.82}$$

and hence to

$$\left((\mathbf{v}\cdot\nabla\;+\;\frac{\partial}{\partial t}\right)\mathcal{D}(\mathbf{r},t) \;=\; 0. \tag{S.83}$$

And indeed, for the $\mathcal{D}$ as in eq. (S.78),

$$\nabla\mathcal{D} \;=\; -2(c^2t-\mathbf{v}\cdot\mathbf{r})\mathbf{v} \;+\; 2(c^2-v^2)\mathbf{r}, \tag{S.84}$$

$$\mathbf{v}\cdot\nabla\mathcal{D} \;=\; -2v^2(c^2t-\mathbf{v}\cdot\mathbf{r}) \;+\; 2(c^2-v^2)(\mathbf{v}\cdot$$

$$)$$

$$=\; 2c^2\big(\mathbf{v}\cdot$$

$-\,v^2t\big)$,(S.85)

$$\frac{\partial\mathcal{D}}{\partial\sqcup} \;=\; +2c^2(c^2t-\mathbf{v}\cdot\mathbf{r}) \;+\; (c^2-v^2)(-2c^2t)$$

$$=\; -2c^2\big(\mathbf{v}\cdot$$

$-\,v^2t\big)$,(S.86)

hence

$$\mathbf{v}\cdot\nabla\mathcal{D} \;+\; \frac{\partial\mathcal{D}}{\partial\sqcup} \;=\; 0 \tag{S.87}$$

exactly as required in eq. (S.83). And as we saw a minute earlier, eq. (S.83) immediately leads to the Landau gauge condition (S.79) for the potentials (S.77).