PHY–352 L. Solutions for problem set #7.

Problem **1**:

For the observer on the z axis, the distance to the particle remains constant

$$\mathcal{R} \;=\; \sqrt{z^2+a^2} \tag{S.1}$$

while the particle keeps moving in the circle. Also, at any retarded time $t_{\rm ret}$, the particle's velocity vector $\mathbf{v}$ is $\perp$ to the direction $\mathbf{n}$ towards the observer, hence

$$1 \,-\, \boldsymbol{\beta}\cdot\mathbf{n} \;\equiv\; 1. \tag{S.2}$$

Consequently, the Liénard–Wiechert potentials (for the observer on z axis) are simply

$$V \;=\; \frac{Q}{4\pi\epsilon_0\mathcal{R}} \;=\; \text{const} \tag{S.3}$$

and

$$\mathbf{A} \;=\; \frac{Q\mu_0}{4\pi\mathcal{R}}\,\mathbf{v}(t_{\rm ret}). \tag{S.4}$$

In Cartesian coordinates, the particle's velocity vector is

$$\mathbf{v}(t) \;=\; -\omega a\sin(\omega t)\hat{\mathbf{x}} \;+\; \omega a\cos(\omega t)\hat{\mathbf{y}} \tag{S.5}$$

hence at the retarded time

$$\mathbf{v}(t_{\rm ret}) \;=\; -\omega a\sin(\omega(t-\tfrac{\mathcal{R}}{c}))\hat{\mathbf{x}} \;+\; \omega a\cos(\omega(t-\tfrac{\mathcal{R}}{c}))\hat{\mathbf{y}} \tag{S.6}$$

and therefore

$$\mathbf{A}(z,t) \;=\; \frac{Q\mu_0\omega a}{4\pi\mathcal{R}}\Big(-\sin(\omega(t-\tfrac{\mathcal{R}}{c}))\hat{\mathbf{x}} \,+\, \cos(\omega(t-\tfrac{\mathcal{R}}{c}))\hat{\mathbf{y}}\Big). \tag{S.7}$$

Problem **2**:

(a) An observer on the x axis and to the left of $x = b$ — and hence to the left of the particle at all times — sees it at the retarded time such that

$$ct - ct_{\text{ret}} = |x - w_x(t_{\text{ret}})| = w_x(t_{\text{ret}}) - x, \tag{S.8}$$

hence

$$ct_{\text{ret}} + w_x(t_{\text{ret}}) = ct + x. \tag{S.9}$$

For the hyperbolic motion (1), this equation becomes

$$ct_{\text{ret}} + \sqrt{(ct_{\text{ret}})^2 + b^2} = ct + x. \tag{S.10}$$

The LHS of this equation is always positive, regardless how negative we take the ct_{ret}, so for a negative RHS this equation has no solutions. Physically, this means that for $ct + x \leq 0$ ($i.e.$, for $t \leq -x/c$), the observer cannot see the moving particle.

(b) Now let $ct + x > 0$ and let's solve eq. (S.10) for the (ct_{ret}). Rewriting eq. (S.10) as

$$\sqrt{(ct_{\text{ret}})^2 + b^2} = (ct + x) - (ct_{\text{ret}}) \tag{S.11}$$

and taking the squares of both sides, we get

$$\cancel{(ct_{\text{ret}})^2} + b^2 = (ct + x)^2 - 2(ct + x) \times (ct_{\text{ret}}) + \cancel{(ct_{\text{ret}})^2}, \tag{S.12}$$

and hence

$$(ct_{\text{ret}}) = \frac{(ct + x)^2 - b^2}{2(ct + x)} = \frac{(ct + x)}{2} - \frac{b^2}{2(ct + x)}. \tag{S.13}$$

As to the particle's location at that retarded time,

$$w_x = \sqrt{(ct_{\text{ret}})^2 + b^2} = (ct + x) - (ct_{\text{ret}}) = \frac{(ct + x)}{2} + \frac{b^2}{2(ct + x)}. \tag{S.14}$$

(c) The Liénard–Wiechert potentials have general form

$$V(\mathbf{r},t) \;=\; \frac{Q}{4\pi\epsilon_0}\left[\frac{1}{\mathcal{R}(1-\boldsymbol{\beta}\cdot\mathbf{n})}\right]_{\text{ret}}, \qquad \mathbf{A}(\mathbf{r},t) \;=\; \frac{Q\mu_0}{4\pi}\left[\frac{\mathbf{v}}{\mathcal{R}(1-\boldsymbol{\beta}\cdot\mathbf{n})}\right]_{\text{ret}}. \tag{S.15}$$

For the particle moving along the x axis according to eq. (1) and the observer on the same x axis to the left of the particle, $\mathbf{n}=-\hat{\mathbf{x}}$ and

$$\big[1-\boldsymbol{\beta}\cdot\mathbf{n}\big]_{\text{ret}} \;=\; 1\;+\;\frac{1}{c}\frac{dw(t_{\text{ret}})}{t_{\text{ret}}} \;=\; 1\;+\;\frac{ct_{\text{ret}}}{\sqrt{(ct_{\text{ret}})^2+b^2}}\,. \tag{S.16}$$

Or in terms of the observer time,

$$\begin{aligned}\big[1-\boldsymbol{\beta}\cdot\mathbf{n}\big]_{\text{ret}} \;&=\; 1\;+\;\frac{(ct+x)^2\;-\;b^2}{2(ct+x)}\bigg/\frac{(ct+x)^2\;+\;b^2}{2(ct+x)}\\ &=\;1\;+\;\frac{(ct+x)^2\;-\;b^2}{(ct+x)^2\;+\;b^2}\;=\;\frac{2(ct+x)^2}{(ct+x)^2\;+\;b^2}\,.\end{aligned} \tag{S.17}$$

Also,

$$\mathcal{R} \;=\; \sqrt{(ct_{\text{ret}})^2+b^2}\;-\;x\;=\;\frac{(ct+x)^2+b^2}{2(ct+x)}\;-\;x=\;\frac{(ct)^2-x^2+b^2}{2(ct+x)}\,, \tag{S.18}$$

hence

$$\big[\mathcal{R}(1-\boldsymbol{\beta}\cdot\mathbf{n})\big]_{\text{ret}} \;=\; (ct+x)\times\frac{(ct)^2-x^2+b^2}{(ct+x)^2+b^2}\,, \tag{S.19}$$

and therefore the scalar potential

$$V(x,t) \;=\; \frac{Q}{4\pi\epsilon_0}\times\frac{(ct+x)^2+b^2}{(ct+x)(c^2t^2-x^2+b^2)}\,. \tag{S.20}$$

Note: the $c^2t^2-x^2+b^2$ factor in the denominator is positive because $-ct<x<b$.

As to the vector potential,

$$\mathbf{A}(x,t) \;=\; \frac{V(x,t)\hat{\mathbf{x}}}{c^2}\,v_x(t_{\rm ret}), \tag{S.21}$$

where the particle's velocity at the retarded time was

$$v_x(t_{\rm ret}) \;=\; \frac{dw_x(t_{\rm ret})}{dt_{\rm ret}} \;=\; \frac{c^2 t_{\rm ret}}{\sqrt{(ct_{\rm ret})^2+b^2}} \;=\; c\,\frac{(ct+x)^2\;-\;b^2}{(ct+x)^2\;+\;b^2}\,, \tag{S.22}$$

hence

$$\mathbf{A}(x,t) \;=\; \frac{Q\,\hat{\mathbf{x}}}{4\pi\epsilon_0 c} * \frac{(ct+x)^2-b^2}{(ct+x)(c^2t^2-x^2+b^2)}\,. \tag{S.23}$$

<ins>Problem **3**</ins>:

(a) Let's start with the general formulae for the **E** and **B** fields of a moving point charge:

$$\mathbf{E}(\mathbf{r},t) = \;\mathbf{E}_C(\mathbf{r},t)\;+\;\mathbf{E}_A(\mathbf{r},t), \tag{S.24}$$

$$\mathbf{E}_C(\mathbf{r},t) \;=\; \frac{Q}{4\pi\epsilon_0}\left[\frac{(1-\beta^2)(\mathbf{n}-\boldsymbol{\beta})}{\mathcal{R}^2(1-\boldsymbol{\beta}\cdot\mathbf{n})^3}\right]_{\rm ret}, \tag{S.25}$$

$$\mathbf{E}_A(\mathbf{r},t) \;=\; \frac{Q\mu_0}{4\pi}\left[\frac{\mathbf{n}\times((\boldsymbol{\beta}-\mathbf{n})\times\mathbf{a})}{\mathcal{R}(1-\boldsymbol{\beta}\cdot\mathbf{n})^3}\right]_{\rm ret}, \tag{S.26}$$

$$\mathbf{B}(\mathbf{r},t) \;=\; \frac{\mathbf{n}_{\rm ret}}{c}\times\mathbf{E}(\mathbf{r},t)\,, \tag{S.27}$$

see my notes on Liénard–Wiechert potentials, eqs. (137) and (145) on pages 26 and 28.

For the particle motion confined to the x axis and the observer lying on the same x axis to the right of the particle, we have

$$\mathbf{n} \;=\; \hat{\mathbf{x}}, \quad \boldsymbol{\beta} \;=\; \beta\hat{\mathbf{x}}, \quad \mathbf{a} \;=\; a\hat{\mathbf{x}}. \tag{S.28}$$

Consequently, the vanishing cross-products $\hat{\mathbf{x}}\times\hat{\mathbf{x}}=\mathbf{0}$ immediately give us

$$\mathbf{E}_C \parallel \hat{\mathbf{x}}, \qquad \text{while} \quad \mathbf{E}_A \;=\; 0 \quad \text{and} \quad \mathbf{B} \;=\; 0. \tag{S.29}$$

Specifically,

$$\mathbf{E}_C = \frac{Q}{4\pi\epsilon_0} * \frac{(1-\beta^2)(\mathbf{n}-\boldsymbol{\beta})}{\mathcal{R}^2(1-\boldsymbol{\beta}\cdot\mathbf{n})^3}, \tag{S.30}$$

where

$$(\mathbf{n}-\boldsymbol{\beta}) = (1-\beta)\hat{\mathbf{x}}, \qquad (1-\boldsymbol{\beta}\cdot\mathbf{n}) = (1-\beta), \tag{S.31}$$

hence

$$\frac{(1-\beta^2)(\mathbf{n}-\boldsymbol{\beta})}{(1-\boldsymbol{\beta}\cdot\mathbf{n})^3} = \frac{(1-\beta^2)(1-\beta)\hat{\mathbf{x}}}{(1-\beta)^3} = \left(\frac{1+\beta}{1-\beta}\right)\hat{\mathbf{x}} = \left(\frac{c+v}{c-v}\right)\hat{\mathbf{x}}, \tag{S.32}$$

and therefore

$$\mathbf{E}(\mathbf{r},t) = \mathbf{E}_C(\mathbf{r},t) = \frac{Q\hat{\mathbf{x}}}{4\pi\epsilon_0}\left[\frac{1}{\mathcal{R}^2}\left(\frac{c+v}{c-v}\right)\right]_{\text{ret}}. \tag{S.33}$$

Quod erat demonstrandum.

(b) For the observer on the x axis but to the left from the particle we have

$$\mathbf{n} = -\hat{\mathbf{x}}, \quad \boldsymbol{\beta} = +\beta\hat{\mathbf{x}}, \quad \mathbf{a} = +a\hat{\mathbf{x}}. \tag{S.34}$$

Consequently, exactly as in part (a) we have

$$\mathbf{E}_C \parallel \hat{\mathbf{x}}, \qquad \text{while} \quad \mathbf{E}_A = 0 \quad \text{and} \quad \mathbf{B} = 0. \tag{S.35}$$

just from the vanishing cross-products. But this time, in the equation

$$\mathbf{E}_C = \frac{Q}{4\pi\epsilon_0} * \frac{(1-\beta^2)(\mathbf{n}-\boldsymbol{\beta})}{\mathcal{R}^2(1-\boldsymbol{\beta}\cdot\mathbf{n})^3}, \tag{S.36}$$

for the generalized Coulomb electric field, we have

$$(\mathbf{n}-\boldsymbol{\beta}) = -(1+\beta)\hat{\mathbf{x}} \quad \text{and} \quad (1-\boldsymbol{\beta}\cdot\mathbf{n}) = (1+\beta),$$

hence

$$\frac{(1-\beta^2)(\mathbf{n}-\boldsymbol{\beta})}{(1-\boldsymbol{\beta}\cdot\mathbf{n})^3} \;=\; -\frac{(1-\beta^2)(1+\beta)\hat{\mathbf{x}}}{(1+\beta)^3} \;=\; -\left(\frac{1-\beta}{1+\beta}\right)\hat{\mathbf{x}} \;=\; \left(\frac{c-v}{c+v}\right)\hat{\mathbf{x}}, \tag{S.37}$$

and therefore

$$\mathbf{E}(\mathbf{r},t) \;=\; \mathbf{E}_C(\mathbf{r},t) \;=\; -\frac{Q\hat{\mathbf{x}}}{4\pi\epsilon_0}\left[\frac{1}{\mathcal{R}^2}\left(\frac{c-v}{c+v}\right)\right]_{\rm ret}. \tag{S.38}$$

Problem **4**:

The EM power flowing through any surface is the flux of the Poynting vector $\mathbf{S}=\mathbf{E}\times\mathbf{H}$ through that surface; for the xy plane, this means

$$P \;=\; \iint dx\,dy\,S_z\,. \tag{S.39}$$

The EM fields generated by the point charge Q moving at constant velocity $\mathbf{v}=v\hat{\mathbf{z}}$ were written down in class,

$$\mathbf{E}(s,z,t) \;=\; \frac{Q}{4\pi\epsilon_0}\,\frac{\gamma((z-vt)\hat{\mathbf{z}}+s\hat{\mathbf{s}})}{[\gamma^2(z-vt)^2+s^2]^{3/2}}\,, \tag{S.40}$$

$$\mathbf{B}(s,z,t) \;=\; \frac{Q\mu_0}{4\pi}\,\frac{\gamma vs\hat{\boldsymbol{\phi}}}{[\gamma^2(z-vt)^2+s^2]^{3/2}}\,, \tag{S.41}$$

cf. my notes, eq. (122) on page 23. The Poynting vector for these fields is

$$\mathbf{S} \;=\; \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B} \;=\; \frac{Q^2}{16\pi^2\epsilon_0}\,\frac{\gamma^2vs[(z-vt)(\hat{\mathbf{z}}\times\hat{\boldsymbol{\phi}}=-\hat{\mathbf{s}})\;+\;s(\hat{\mathbf{s}}\times\hat{\boldsymbol{\phi}}=\hat{\mathbf{z}})]}{[\gamma^2(z-vt)^2+s^2]^3}\,, \tag{S.42}$$

so focusing on its z component

$$S_z \;=\; \frac{Q^2}{16\pi^2\epsilon_0}\,\frac{\gamma^2vs^2}{[\gamma^2(z-vt)^2+s^2]^3}\,, \tag{S.43}$$

we find the net power flux through the xy plane as the integral

$$P \;=\; \frac{Q^2\gamma^2v}{16\pi^2\epsilon_0}\iint\frac{s^2}{[\gamma^2(z_{\rm plane}-vt)^2+s^2]^3}\,d^2{\rm area}\,. \tag{S.44}$$

In this formula, $z_{\rm plane}=0$ (it's the z coordinate of the xy plane), while the time t is chosen such that the particle itself is at $w_z = vt = -a$, thus

$$P = \frac{Q^2\gamma^2 v}{16\pi^2\epsilon_0} \iint \frac{s^2}{[\gamma^2 a^2 + s^2]^3}\, d^2\text{area}\,. \tag{S.45}$$

To evaluate the remaining integral here, we go to the polar coordinates (s, ϕ) for the xy plane, thus

$$d^2\text{area} = s\,ds\,d\phi \to 2\pi s\,ds \tag{S.46}$$

since the integrand does not depend on ϕ. Consequently,

$$\begin{aligned}
\iint \frac{s^2}{[\gamma^2 a^2 + s^2]^3}\, d^2\text{area} &= \int\limits_0^\infty \frac{s^2}{[\gamma^2 a^2 + s^2]^3}\, 2\pi s\, ds \\
&\langle\!\langle \text{ changing integration variable from } s \text{ to } \nu = \gamma^2 a^2 + s^2,\ 2\pi s\, s = \pi\, d\nu \,\rangle\!\rangle \\
&= \int\limits_{\gamma^2 a^2}^\infty \frac{(\nu - \gamma^2 a^2)}{\nu^3}\, \pi d\nu \\
&= \pi\left(-\frac{1}{\nu} + \frac{\gamma^2 a^2}{2\nu^2}\right)\Bigg|_{\nu=\gamma^2\nu^2}^{\nu=\infty} \\
&= 0 - \pi\left(-\frac{1}{\gamma^2 a^2} + \frac{\gamma^2 a^2}{2(\gamma^2 a^2)^2}\right) \\
&= +\frac{\pi}{2\gamma^2 a^2}\,,
\end{aligned} \tag{S.47}$$

and therefore

$$P = \frac{Q^2 v}{32\pi\epsilon_0 a^2}\,. \tag{S.48}$$

Problem **5**:

(a) The charge Q_1 is at rest, so it's electric field is purely Coulomb while its magnetic field is absent altogether,

$$\mathbf{E}_1(\mathbf{r},t) \;=\; \frac{Q_1\mathbf{n}}{4\pi\epsilon_0\,\mathbf{r}^2}\,, \qquad \mathbf{B}_1 \;\equiv\; \mathbf{0}. \tag{S.49}$$

Consequently, the force on the second charge Q_2 is

$$\mathbf{F}_{12} \;=\; Q_2\mathbf{E}_1(\mathbf{r}_2(t),t) \;+\; Q_2\mathbf{v}_2\times\mathbf{B}_1(\mathbf{r}_2(t),t) \;=\; Q_2\mathbf{E}_1(\mathbf{r}_2(t),t), \tag{S.50}$$

which depends only on the relative positions of the two charges. Specifically, Q_1 is stuck at the coordinate origin while Q_2 is on the z axis at $z = vt$. Assuming $z = vt > 0$, we have $\mathbf{n} = +\hat{\mathbf{z}}$ and hence

$$\mathbf{F}_{12}(t) \;=\; +\frac{Q_1Q_2\hat{\mathbf{z}}}{4\pi\epsilon_0}\,\frac{1}{(vt)^2}\,. \tag{S.51}$$

(b) The second charge Q_2 moves at a constant velocity $\mathbf{v} = v\hat{\mathbf{z}}$ along the z axis, so the EM fields it generates are

$$\mathbf{E}_2(s,z,t) \;=\; \frac{Q_2}{4\pi\epsilon_0}\,\frac{\gamma((z-vt)\hat{\mathbf{z}}+s\hat{\mathbf{s}})}{[\gamma^2(z-vt)^2+s^2]^{3/2}}\,, \tag{S.52}$$

$$\mathbf{B}_2(s,z,t) \;=\; \frac{Q_2\mu_0}{4\pi}\,\frac{\gamma vs\hat{\boldsymbol{\phi}}}{[\gamma^2(z-vt)^2+s^2]^{3/2}}\,, \tag{S.53}$$

cf. my notes, eq. (122) on page 23. Since the first charge is at rest at the origin, the EM force on it is simply

$$\mathbf{F}_{21} \;=\; Q_1\mathbf{E}_2(\mathbf{0},t) \;=\; \frac{Q_1Q_2}{4\pi\epsilon_0}\,\frac{\gamma(-vt)\hat{\mathbf{z}}}{[\gamma^2(vt)^2]^{3/2}} \;=\; -\frac{Q_1Q_2\hat{\mathbf{z}}}{4\pi\epsilon_0}\,\frac{1}{\gamma^2(vt)^2}\,. \tag{S.54}$$

Note: $\mathbf{F}_{21} \neq \mathbf{F}_{12}$! Instead, the $\mathbf{F}_{21}$ is smaller by the factor $\gamma^{-2} = (1-\beta^2)$.

(c) The net momentum of the EM fields obtains a volume integral of the Poynting vector, or rather of

$$\mathbf{g} \;=\; \mathbf{D}\times\mathbf{B} \;=\; \epsilon_0\mathbf{E}\times\mathbf{B} \;=\; \frac{1}{c^2}\mathbf{S}, \tag{S.55}$$

$$\mathbf{p}_{\rm em} \;=\; \iiint\limits_{\substack{\rm whole\\ \rm space}} \epsilon_0\mathbf{E}\times\mathbf{B}\,d^3{\rm Vol}. \tag{S.56}$$

For the EM fields generated by the two particles,

$$\begin{aligned} \mathbf{E}\times\mathbf{B} \;&=\; (\mathbf{E}_1+\mathbf{E}_2)\times(\mathbf{B}_1+\mathbf{B}_2)\\ \langle\!\langle\,\text{since } \mathbf{B}_1=0\,\rangle\!\rangle \qquad &=\; \mathbf{E}_1\times\mathbf{B}_2 \;+\; \mathbf{E}_2\times\mathbf{B}_2\,. \end{aligned} \tag{S.57}$$

Consequently, we may split the net EM momentum into two terms,

$$\begin{aligned} \mathbf{p}_{\rm em} \;&=\; \mathbf{p}_{12} \;+\; \mathbf{p}_{22}\,,\\ \mathbf{p}_{12} \;&=\; \iiint \epsilon_0\mathbf{E}_1\times\mathbf{B}_2\,d^3{\rm Vol},\\ \mathbf{p}_{22} \;&=\; \iiint \epsilon_0\mathbf{E}_2\times\mathbf{B}_2\,d^3{\rm Vol}. \end{aligned} \tag{S.58}$$

However, the $\mathbf{p}_{22}$ momentum here is time-independent. Indeed, both the electric field $\mathbf{E}_2$ and the magnetic field $\mathbf{B}_2$ generated by the moving charge move along with that charge at a constant velocity $v\hat{\mathbf{z}}$. Consequently, the integrals (S.58) for the $\mathbf{p}_{22}$ evaluated at any two different times t_1 and t_2 are related by a simple shift of the integration variable, $z_2 = z_1 + (t_2 - t_1)v$. Thus, once we integrate over the whole space, we end up with equal integrals, $\mathbf{p}_{22}(t_2) = \mathbf{p}_{22}(t_1)$.

For our purposes, we are interested in the time derivative of the net EM momentum $\mathbf{p}_{\rm em}(t)$, so we do not care about time-independent terms like the $\mathbf{p}_{22}$. Instead, we simply write

$$\mathbf{p}_{em}(t) \;=\; \mathbf{p}_{12}(t) \;+\; \text{const} \tag{S.59}$$

and don't bother to evaluate the integral for the $\mathbf{p}_{22}$.

Now comes the hard part: calculating the integral for the time-dependent $\mathbf{p}_{12}(t)$. Given the fields (S.49) and (S.53), we have

$$\begin{aligned}\epsilon_0\mathbf{E}_1\times\mathbf{B}_2 &= \frac{Q_1Q_2\mu_0}{16\pi^2}\,\frac{(s\hat{\mathbf{s}}+z\hat{\mathbf{z}})\times(\gamma vs\hat{\boldsymbol{\phi}})}{[s^2+z^2]^{3/2}[s^2+(z-vt)^2]^{3/2}}\\ &= \frac{Q_1Q_2\mu_0\gamma v}{16\pi^2}*\frac{s^2\hat{\mathbf{z}}\,-\,sz\hat{\mathbf{s}}}{[s^2+z^2]^{3/2}[s^2+(z-vt)^2]^{3/2}}\,,\end{aligned}\tag{S.60}$$

hence

$$\begin{aligned}\mathbf{p}_{12} &= \frac{Q_1Q_2\mu_0\gamma v}{16\pi^2}\iiint\frac{s^2\hat{\mathbf{z}}\,-\,sz\hat{\mathbf{s}}}{[s^2+z^2]^{3/2}[s^2+(z-vt)^2]^{3/2}}\,d^3\mathrm{Vol}\\ &= \frac{Q_1Q_2\mu_0\gamma v}{16\pi^2}\int\limits_{-\infty}^{+\infty}dz\int\limits_0^{\infty}s\,ds\int\limits_0^{2\pi}d\phi\,\frac{s^2\hat{\mathbf{z}}\,-\,sz\hat{\mathbf{s}}}{[s^2+z^2]^{3/2}[s^2+(z-vt)^2]^{3/2}}\,.\end{aligned}\tag{S.61}$$

To take this 3D integral, we start by integrating over the ϕ. By the rotational symmetry (around the z axis),

$$\int\limits_0^{2\pi}d\phi\,\hat{\mathbf{z}} = 2\pi\hat{\mathbf{z}},\qquad \int\limits_0^{2\pi}d\phi\,\hat{\mathbf{s}} = \mathbf{0},\tag{S.62}$$

hence

$$\int\limits_0^{2\pi}d\phi\,\frac{s^2\hat{\mathbf{z}}\,-\,sz\hat{\mathbf{s}}}{\phi\text{-independent constant}} = \frac{2\pi s^2\hat{\mathbf{z}}}{\text{same constant}}\,,\tag{S.63}$$

and therefore

$$\mathbf{p}_{12} = \frac{Q_1Q_2\mu_0\gamma v\,\hat{\mathbf{z}}}{8\pi}\int\limits_{-\infty}^{+\infty}dz\int\limits_0^{\infty}\frac{s^3\,ds}{[s^2+z^2]^{3/2}[s^2+(z-vt)^2]^{3/2}}\,.\tag{S.64}$$

Next, the s integral here follows from eq. (5) for $a=|z|$ and $b=\gamma|z-vt|$, thus

$$\int\limits_0^{\infty}\frac{s^3\,ds}{[s^2+z^2]^{3/2}[s^2+(z-vt)^2]^{3/2}} = \frac{1}{(|z|+\gamma|z-vt|)^2}\tag{S.65}$$

and hence

$$\mathbf{p}_{12} \;=\; \frac{Q_1 Q_2 \mu_0 \gamma v\,\hat{\mathbf{z}}}{8\pi} \int\limits_{-\infty}^{+\infty} \frac{dz}{(|z| + \gamma|z - vt|)^2}\,. \tag{S.66}$$

In the denominator inside the integral here

$$|z| \,+\, \gamma|z - vt| \;=\; \begin{cases} \gamma vt \,-\, (\gamma+1)z & \text{for } z < 0, \\ \gamma vt \,-\, (\gamma-1)z & \text{for } 0 < z < vt, \\ -\gamma vt \,+\, (\gamma+1)z & \text{for } z > vt, \end{cases} \tag{S.67}$$

hence

$$\begin{aligned} \int\limits_{-\infty}^{+\infty} \frac{dz}{(|z| + \gamma|z - vt|)^2} \;&=\; \int\limits_{-\infty}^{0} \frac{dz}{[\gamma vt - (\gamma+1)z]^2} \,+\, \int\limits_{0}^{vt} \frac{dz}{[\gamma vt - (\gamma-1)z]^2} \\ &\qquad +\, \int\limits_{vt}^{+\infty} \frac{dz}{[(\gamma+1)z - \gamma vt]^2} \\ &=\; \left.\frac{1}{(\gamma+1)[\gamma vt - (\gamma+1)z]}\right|_{z=-\infty}^{z=0} \,+\, \left.\frac{1}{(\gamma-1)[\gamma vt - (\gamma-1)z]}\right|_{z=0}^{z=vt} \\ &\qquad +\, \left.\frac{-1}{(\gamma+1)[(\gamma+1)z - \gamma vt]}\right|_{z=vt}^{z=+\infty} \\ &=\; \frac{1}{(\gamma+1)\gamma vt} \,+\, \frac{1}{\gamma-1}\left(\frac{1}{vt} - \frac{1}{\gamma vt}\right) + \frac{1}{(\gamma+1)vt} \\ &=\; \frac{2}{\gamma vt}\,. \end{aligned} \tag{S.68}$$

Plugging this formula into eq. (S.66), we arrive at

$$\mathbf{p}_{12} \;=\; \frac{Q_1 Q_2 \mu_0 \gamma v\,\hat{\mathbf{z}}}{8\pi} * \frac{2}{\gamma vt} \;=\; \frac{Q_1 Q_2 \mu_0\,\hat{\mathbf{z}}}{4\pi\, t} \tag{S.69}$$

and hence

$$\mathbf{p}_{\rm em}(t) \;=\; \frac{Q_1 Q_2 \mu_0\,\hat{\mathbf{z}}}{4\pi\, t} \,+\, \text{const.} \tag{4}$$

Quod erat demonstrandum.

(d) In light of parts (a) and (b)

$$\mathbf{F}_{12} + \mathbf{F}_{21} = \frac{Q_1Q_2\,\hat{\mathbf{z}}}{4\pi\epsilon_0(vt)^2}\left(1 - \frac{1}{\gamma^2} = \beta^2 = \frac{v^2}{c^2}\right) = \frac{Q_1Q_2\mu_0}{4\pi t^2}\,. \tag{S.70}$$

At the same time, for the EM momentum (4) we derived in part (c),

$$\frac{d\mathbf{p}_{\rm em}}{dt} = -\frac{Q_1Q_2\mu_0\,\hat{\mathbf{z}}}{4\pi\,t^2}, \tag{S.71}$$

so altogether

$$\mathbf{F}_{12} + \mathbf{F}_{21} + \frac{d\mathbf{p}_{\rm em}}{dt} = 0. \tag{3}$$

Quod erat demonstrandum.

Problem **6**:

The proton falls down through height $h = 1$ cm in time $t = \sqrt{2h/g} \approx 0.045$ s. During this time, its acceleration g makes it radiate power

$$P = \frac{e^2\mu_0}{6\pi c} \times g^2 \approx 5.5 \cdot 10^{-52}\ \mathrm{W}, \tag{S.72}$$

so the net energy it radiate while falling the first centimeter down is

$$U_{\rm rad} = Pt \approx 2.5 \cdot 10^{-53}\ \mathrm{J}. \tag{S.73}$$

At the same time, the proton's potential energy loss while it falls through that centimeter is

$$|\Delta U_{\rm pot}| = m_p gh = 1.64 \cdot 10^{-28}\ \mathrm{J}, \tag{S.74}$$

thus

$$\frac{U_{\rm rad}}{|\Delta U_{\rm pot}|} \approx 1.5 \cdot 10^{-25}, \tag{S.75}$$

a *very* tiny fraction.

Problem **7**:

An accelerating non-relativistic particle radiates EM energy at the rate

$$P = \frac{q^2\mu_0}{6\pi c} \times a^2, \tag{S.76}$$

so the total energy it radiates as it scatters off the stationary charge Q is

$$\begin{aligned} U_{\text{rad}} &= \int P(t)\,dt = \frac{q^2\mu_0}{6\pi c}\int a^2(t)\,dt \\ &= \frac{q^2\mu_0}{6\pi c}\int a(t)\,dv(t) = \frac{q^2\mu_0}{6\pi c}\int a(v)\,dv, \end{aligned} \tag{S.77}$$

where the range of the integrals is $-\infty < t < +\infty$ for the time, or $-v_0 < v < +v_0$ for the velocity, thus

$$U_{\text{rad}} = \frac{q^2\mu_0}{6\pi c}\int_{-v_0}^{+v_0} a(v)\,dv. \tag{S.78}$$

To evaluate this integral, we do not need to know how the velocity and the acceleration change with time as the particle flies in and out; all we need is to relate the acceleration to the velocity at the same time.

To find this relation $a(v)$, we start with the Newton Second Law

$$ma = F_{\text{Coulomb}} = \frac{Qq}{4\pi\epsilon_0} \times \frac{1}{x^2} \tag{S.79}$$

(where x is the distance to the stationary charge Q) and the energy conservation equation

$$\tfrac{1}{2}mv^2 + \frac{Qq}{4\pi\epsilon_0} \times \frac{1}{x} = \tfrac{1}{2}mv_0^2\,. \tag{S.80}$$

Consequently, given a velocity v (between $-v_0$ and $+v_0$), we can say that the particle had this velocity at the point x where

$$\frac{1}{x} = \frac{m(2\pi\epsilon_0)(v_0^2 - v^2)}{Qq} \tag{S.81}$$

so the acceleration at that point was

$$a = \frac{Qq}{4\pi\epsilon_0 m} \times \frac{1}{x^2} = \frac{\pi\epsilon_0 m}{Qq} \times (v_0^2 - v^2)^2. \tag{S.82}$$

Given eq. (S.82) for the acceleration as a function of the velocity, the radiated energy (S.78) obtains as a simple integral

$$\begin{aligned} U_{\mathrm{rad}} &= \frac{q^2\mu_0}{6\pi c}\int\limits_{-v_0}^{+v_0} a(v)\,dv \\ &= \frac{q^2\mu_0}{6\pi c}\times\frac{\pi\epsilon_0 m}{Qq}\times\int\limits_{-v_0}^{+v_0}(v_0^2-v^2)^2\,dv \\ &= \frac{q}{Q}\,\frac{m}{6c^3}\times\frac{16}{15}\,v_0^5 \\ &= \frac{16}{45}\,\frac{q}{Q}\times\frac{mv_0^2}{2}\times\frac{v_0^3}{c^3}\,. \end{aligned} \tag{S.83}$$

In other words, as a fraction of the initial kinetic energy $U_0=\frac{1}{2}mv_0^2$, the energy lost to the EM radiation is

$$\frac{U_{\mathrm{rad}}}{U_0} = \frac{16}{45}\,\frac{q}{Q}\left(\frac{v_0}{c}\right)^3 \tag{S.84}$$

which is presumably a rather small fraction since $v_0 \ll c$.

Problem **8** is postponed.