PHY–352 L. Solutions for problem set #8.

Problem **1**:

(a) By the translational symmetry in x and y directions, the vector potential $\mathbf{A}(x,y,z,t)$ does not depend on x and y but only on z and t. Specifically, the retarded potential is

$$\mathbf{A}(z,t) \;=\; \frac{\mu_0\hat{\mathbf{x}}}{4\pi}\iint dx'\,dy'\,\frac{K(t-\mathcal{R}/c)}{\mathcal{R}} \tag{S.1}$$

where

$$\mathcal{R} \;=\; \sqrt{(x'-x)^2+(y'-y)^2+z^2}\,. \tag{S.2}$$

Now let's change the Cartesian coordinates (x',y') for the current sheet to the polar coordinates (s,ϕ) centered at the (x,y), thus

$$x' \;=\; x\;+\;s\times\cos\phi,\quad y' \;=\; y\;+\;s\times\sin\phi\quad \mathcal{R} \;=\; \sqrt{s^2+z^2}. \tag{S.3}$$

For these coordinates

$$dx'\,dy' \;=\; s\,ds\,d\phi \;\to\; 2\pi s\,ds, \tag{S.4}$$

hence

$$\mathbf{A}(z,t) \;=\; \frac{\mu_0\hat{\mathbf{x}}}{4\pi}\int\limits_0^\infty \frac{K(t-\mathcal{R}/c)}{\mathcal{R}}\,2\pi s\,ds. \tag{S.5}$$

But

$$2\pi s\,ds \;=\; \pi d(s^2) \;=\; \pi d(\mathcal{R}^2=s^2+z^2) \;=\; 2\pi\mathcal{R}\,d\mathcal{R}, \tag{S.6}$$

so changing the integration variable from s to $\mathcal{R}$, we get

$$\mathbf{A}(z,t) \;=\; \frac{\mu_0\hat{\mathbf{x}}}{4\pi}\int\limits_z^\infty K(t-rr/c)\,2\pi d\mathcal{R}. \tag{S.7}$$

Finally, to bring this integral to the form (1) we change the integration variable once again, from $\mathcal{R}$ to $u = (\mathcal{R} - z)/c$, thus

$$t - \frac{\mathcal{R}}{c} = t - \frac{z}{c} - u, \qquad d\mathcal{R} = c\,du, \tag{S.8}$$

so the integral (S.7) becomes

$$\mathbf{A}(z,t) = \frac{\mu_0 c}{2}\,\hat{\mathbf{x}} \int_0^\infty K(t - (z/c) - u)\,du. \tag{1}$$

Note: as written, eq. (1) applies only for $z > 0$, *i.e.* above the current sheet. Below the current sheet, we have a similar formula but with $-z$ in place of z. Or for both sides of the current sheet,

$$\mathbf{A}(z,t) = \frac{\mu_0 c}{2}\,\hat{\mathbf{x}} \int_0^\infty K\bigl(t - \tfrac{|z|}{c} - u\bigr)\,du. \tag{S.9}$$

(b) The current sheet does not generate any scalar potential at all, only the vector potential, so the electric field obtains as

$$\mathbf{E}(z,t) = -\frac{\partial \mathbf{A}}{\partial t} = -\frac{\mu_0 c}{2}\,\hat{\mathbf{x}} \int_0^\infty \frac{\partial K(t - \frac{|z|}{c} - u)}{\partial t}\,du. \tag{S.10}$$

But

$$\frac{\partial}{\partial t} K(t - \tfrac{|z|}{c} - u) = -\frac{\partial}{\partial u} K(t - \tfrac{|z|}{c} - u), \tag{S.11}$$

hence

$$\begin{aligned} \int_0^\infty \frac{\partial K(t - \frac{|z|}{c} - u)}{\partial t}\,du &= -\int_0^\infty \frac{\partial K(t - \frac{|z|}{c} - u)}{\partial u}\,du \\ &= -K(t - \tfrac{|z|}{c} - \infty) + K(t - \tfrac{|z|}{c} - 0) \\ &= 0 + K(t - \tfrac{|z|}{c}), \end{aligned} \tag{S.12}$$

and therefore

$$\mathbf{E}(z,t) = -\frac{\mu_0 c}{2}\,\hat{\mathbf{x}}\,K\bigl(t - \tfrac{|z|}{c}\bigr) \tag{S.13}$$

Next, the magnetic field

$$\mathbf{B}(z,t) \;=\; \nabla\times\mathbf{A}(z,t) \;=\; \frac{\mu_0 c}{2}\bigl(\hat{\mathbf{z}}\times\hat{\mathbf{x}} \;=\; \hat{\mathbf{y}}\bigr)\int\limits_0^\infty \frac{\partial K(t-\frac{|z|}{c}-u)}{\partial z}\,du. \tag{S.14}$$

But

$$\frac{\partial}{\partial z}K(t-\tfrac{|z|}{c}-u) \;=\; \frac{\operatorname{sign}(z)}{c}\,\frac{\partial}{\partial u}K(t-\tfrac{|z|}{c}-u), \tag{S.15}$$

hence

$$\begin{aligned}\mathbf{B}(z,t) \;&=\; \frac{\mu_0 c}{2}\,\hat{\mathbf{y}}\,\frac{\operatorname{sign}(z)}{c}\int\limits_0^\infty \frac{\partial K(t-\frac{|z|}{c}-u)}{\partial u}\,du\\ &=\; \frac{\mu_0\operatorname{sign}(z)}{2}\,\hat{\mathbf{y}}\Bigl[K(t-\tfrac{|z|}{c}-\infty)\;-\;K(t-\tfrac{|z|}{c}-0)\Bigr]\\ &=\; -\frac{\mu_0\operatorname{sign}(z)}{2}\,\hat{\mathbf{y}}\,K\bigl(t-\tfrac{|z|}{c}\bigr).\end{aligned} \tag{S.16}$$

Finally, the Poynting vector

$$\mathbf{S}(z,t) \;=\; \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B} \;=\; \frac{\mu_0 c\operatorname{sign}(z)}{4}\bigl(\hat{\mathbf{x}}\times\hat{\mathbf{y}} \;=\; \hat{\mathbf{z}}\bigr)*K^2\bigl(t-\tfrac{|z|}{c}\bigr) \;=\; \frac{\mu_0 c}{4}\,K^2\bigl(t-\tfrac{|z|}{c}\bigr)*\operatorname{sign}(z)\hat{\mathbf{z}}. \tag{S.17}$$

Note the direction of this Poynting vector: above the current sheet it points up while below the current sheet it points down, thus in both cases the energy flows away from the current sheet.

(c) Above the current sheet, the energy flows up with density (S.17), and below the current sheet it flows down with the same density. Measuring this energy flow at the delayed time $t_d = t_0 + \frac{|z|}{c}$ at a height $+z$ above the sheet and $-z$ below the sheet, we have

$$\begin{aligned}\frac{\text{net flow at time } t_0}{\text{unit of area}} \;&=\; +S_z(t_d,+z)\;-\;S_z(t_d,-z)\\ &=\; +\frac{\mu_0 c}{4}K^2(t_d-\tfrac{|z|}{c}=t_0)\;-\;\frac{-\mu_0 c}{4}K^2(t_d-\tfrac{|z|}{c}=t_0)\\ &=\; \frac{\mu_0 c}{2}\,K^2(t_0).\end{aligned} \tag{2}$$

Quod erat demonstrandum.

Problem **2**:

A harmonic oscillator of amplitude A — *i.e.*, $x(t)=A\sin(\omega t)$ — has net energy

$$\begin{aligned} U_{\text{net}} &= U_{\text{pot}} + U_{\text{kin}} \\ &= \frac{M\omega^2}{2}\times\bigl(x=A\sin(\omega t)\bigr)^2 + \frac{M}{2}\times\bigl(\dot{x}=A\omega\cos(\omega t)\bigr)^2 \\ &= \frac{M\omega^2A^2}{2}\bigl(\sin^2(\omega t)+\cos^2(\omega t)\bigr) \\ &= \frac{M\omega^2A^2}{2} = \text{const.} \end{aligned} \tag{S.18}$$

When the oscillating particle is charged, it radiates, and hence slowly loses its energy to radiation. The net radiated power obtains from the Larmor formula

$$P = \frac{Q^2\mu_0}{6\pi c}\times\bigl(\ddot{x} = -\omega^2A\cos(\omega t)\bigr)^2 = \frac{Q^2\mu_0\omega^4A^2}{6\pi c}\times\cos^2(\omega t) \tag{S.19}$$

by time-averaging over a period of oscillation: using $\bigl\langle\cos^2(\omega t)\bigr\rangle=\frac{1}{2}$, we get

$$\langle P\rangle = \frac{Q^2\mu_0\omega^4A^2}{12\pi c}\,. \tag{S.20}$$

By energy conservation, this is the rate at which the oscillator loses its energy,

$$\frac{dU}{dt} = -\langle P\rangle\,, \tag{S.21}$$

hence

$$\begin{aligned} \gamma &\stackrel{\text{def}}{=} -\frac{d\ln(U)}{dt} = -\frac{1}{U}\,\frac{dU}{dt} = +\frac{P}{U} \\ &= \frac{Q^2\mu_0\omega^4A^2}{12\pi c}\bigg/\frac{M\omega^2A^2}{2} \\ &= \frac{Q^2\mu_0\omega^2}{6\pi Mc}\,. \end{aligned} \tag{S.22}$$

Note that the bottom line here is time-independent, so on the top line $\ln(U(t))$ is a linear function of time with the slope $-\gamma$, thus

$$\ln(U(t)) \;=\; \ln(U_0) \;-\; \gamma t \tag{S.23}$$

and therefore

$$U(t) \;=\; U_0 \exp(-\gamma t). \tag{S.24}$$

The oscillation energy is proportional to the amplitude squared, so by the time the amplitude has shrunk to $\frac{1}{2}$ of its initial value, the energy has decreased 4-fold. In light of eq. (S.24), this takes time t such that $\gamma t = \ln(4)$, thus

$$t \;=\; \frac{\ln(4)}{\gamma} \;=\; \frac{6\pi \ln(4)\, Mc}{Q^2 \mu_0 \omega^2}\,. \tag{S.25}$$

For example, a proton of mass $M = 1.673 \cdot 10^{-27}$ kg and charge $Q = 1.602 \cdot 10^{-19}$ C such in a trap where it oscillates with frequency $\omega = 2\pi \times 10^{12}$ Hz would lose half of its oscillation amplitude in time

$$t \;=\; 6\pi \ln(4) \times \frac{(1.673 \cdot 10^{-27}\ \text{kg})(2.998 \cdot 10^{8}\ \text{m/s})}{(1.602 \cdot 10^{-19}\ \text{C})^2 (4\pi \cdot 10^{-7}\ \text{H/m})(2\pi \cdot 10^{12}\ \text{s}^{-1})^2} \;=\; 10.3\ \text{s}. \tag{S.26}$$

Problem **3**:

(a) A time $t = 0$, the ring's electric dipole moment points in the $\hat{\mathbf{x}}$ direction, and its magnitude is

$$p_0 \;=\; \int_0^{2\pi} (x = b\cos\phi) \times (dQ = \lambda b\, d\phi = \lambda_0 \cos(\phi) b\, d\phi) \;=\; \lambda_0 b^2 \int_0^{2\pi} \cos^2\phi\, d\phi \;=\; \lambda_0 b^2 \times \pi. \tag{S.27}$$

As the ring rotates around the z axis with angular velocity ω, the dipole moment rotates in the (x, y) plane, thus

$$\begin{aligned} p_x(t) \;&=\; p_0 \cos(\omega t), \\ p_y(t) \;&=\; p_0 \sin(\omega t), \\ p_z(t) \;&\equiv\; 0. \end{aligned} \tag{S.28}$$

Treating the $p_x(t)$ and the $p_y(t)$ as two separate oscillating dipoles, we see that they have the same frequency ω, the same (real) amplitude $p_0 = \pi \lambda_0 b^2$, but their phases differ by 90°.

(b) Let's focus on the radiation fields in the non-relativistic limit. The radiation fields due to the p_x dipole are

$$\begin{aligned} \mathbf{E}_{\text{rad}}[x] &= \frac{\mu_0(\ddot{p}_x)}{4\pi r}(\mathbf{n}\times(\mathbf{n}\times\hat{\mathbf{x}})) &&= -\frac{p_0\omega^2\mu_0}{4\pi r}\cos(\omega t_{\text{ret}})(\mathbf{n}\times(\mathbf{n}\times\hat{\mathbf{x}})), \\ \mathbf{B}_{\text{rad}}[x] &= -\frac{\mu_0(\ddot{p}_x)^2}{4\pi rc}(\mathbf{n}\times\hat{\mathbf{x}}) &&= +\frac{p_0\omega^2\mu_0}{4\pi rc}\cos(\omega t_{\text{ret}})(\mathbf{n}\times\hat{\mathbf{x}}), \end{aligned} \tag{S.29}$$

while the radiation fields due to the p_y dipole are

$$\begin{aligned} \mathbf{E}_{\text{rad}}[y] &= \frac{\mu_0(\ddot{p}_y)}{4\pi r}(\mathbf{n}\times(\mathbf{n}\times\hat{\mathbf{y}})) &&= -\frac{p_0\omega^2\mu_0}{4\pi r}\sin(\omega t_{\text{ret}})(\mathbf{n}\times(\mathbf{n}\times\hat{\mathbf{y}})), \\ \mathbf{B}_{\text{rad}}[y] &= -\frac{\mu_0(\ddot{p}_y)^2}{4\pi rc}(\mathbf{n}\times\hat{\mathbf{y}}) &&= +\frac{p_0\omega^2\mu_0}{4\pi rc}\sin(\omega t_{\text{ret}})(\mathbf{n}\times\hat{\mathbf{y}}). \end{aligned} \tag{S.30}$$

Adding them up, we get

$$\begin{aligned} \mathbf{E}_{\text{rad}} &= -\frac{p_0\omega^2\mu_0}{4\pi r}\,\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\nu}), \\ \mathbf{B}_{\text{rad}} &= +\frac{p_o\omega^2\mu_0}{4\pi rc}\,\mathbf{n}\times\boldsymbol{\nu} \end{aligned} \tag{S.31}$$

for

$$\boldsymbol{\nu} \stackrel{\text{def}}{=} \cos(\omega t_{\text{ret}})\hat{\mathbf{x}} + \sin(\omega t_{\text{ret}})\hat{\mathbf{y}}\,. \tag{S.32}$$

The Poynting vector for the fields (S.31) is

$$\mathbf{S}_{\text{rad}} = \frac{1}{\mu_0}\mathbf{E}_{\text{rad}}\times\mathbf{B}_{\text{rad}} = -\frac{p_0^2\omega^4\mu_0}{16\pi^2r^2c}\,(\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\nu}))\times(\mathbf{n}\times\boldsymbol{\nu}) \tag{S.33}$$

where

$$\begin{aligned} -(\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\nu}))\times(\mathbf{n}\times\boldsymbol{\nu}) &= +\mathbf{n}(\mathbf{n}\times\boldsymbol{\nu})^2 - (\mathbf{n}\times\boldsymbol{\nu})\big(\mathbf{n}\cdot(\mathbf{n}\times\boldsymbol{\nu}) = 0\big) \\ &= +\mathbf{n}(\mathbf{n}\times\boldsymbol{\nu})^2 \\ &= \mathbf{n}\big[\boldsymbol{\nu}^2 - (\mathbf{n}\cdot\boldsymbol{\nu})^2\big]\,. \end{aligned} \tag{S.34}$$

By construction (S.32) of the vector $\boldsymbol{\nu}$, we have $\boldsymbol{\nu}^2 = 1$ while

$$\mathbf{n}\cdot\boldsymbol{\nu} = \sin\theta[\cos(\phi)\cos(\omega t_{\text{ret}}) + \sin(\phi)\sin(\omega t_{\text{ret}})] = \sin\theta\cos(\omega t_{\text{ret}} - \phi), \tag{S.35}$$

hence

$$\mathbf{S}_{\rm rad} \;=\; +\frac{p_0^2\omega^4\mu_0}{16\pi^2 c}\,\frac{\mathbf{n}}{r^2}*\bigl[1\,-\,\sin^2\theta\cos^2(\omega t_{\rm ret}-\phi)\bigr]. \qquad\text{(S.36)}$$

Time-averaging this Poynting vector over the oscillation period, we get

$$\bigl\langle\cos^2(\omega t_{\rm ret}-\phi)\bigr\rangle \;=\; \bigl\langle\cos^2(\omega t+{\rm const})\bigr\rangle \;=\; \tfrac{1}{2}, \qquad\text{(S.37)}$$

hence

$$\langle\mathbf{S}\rangle_{\rm rad} \;=\; +\frac{p_0^2\omega^4\mu_0}{16\pi^2 c}\,\frac{\mathbf{n}}{r^2}*(1-\tfrac{1}{2}\sin^2\theta). \qquad\text{(S.38)}$$

Finally, dotting this Poynting vector with the area $R^2\mathbf{n}\,d\Omega$ of a segment of a distant sphere, we find the EM power radiated by this oscillator per unit of solid angle,

$$\frac{dP}{d\Omega} \;=\; \frac{p_0^2\omega^4\mu_0}{16\pi^2 c}*\left(1\,-\,\tfrac{1}{2}\sin^2\theta\;=\;\frac{1+\cos^2\theta}{2}\right). \qquad\text{(S.39)}$$

A way to understand this angular power distribution to to add up the powers emitted by the p_x oscillator and by the p_y oscillator. In general the powers do not simply add up, and there is also an interference term, but since the p_x and the p_y oscillators have their phases differ by precisely 90°, the interference term is purely reactive: it makes a non-zero contribution to the instantaneous radiation power, but it time-averages to zero. Consequently, for the situation at hand we do have the two oscillators' powers adding up,

$$\left(\frac{dP}{d\Omega}\right)_{\rm net} \;=\; \left(\frac{dP}{d\Omega}\right)_{p_x}\;+\;\left(\frac{dP}{d\Omega}\right)_{p_y}. \qquad\text{(S.40)}$$

For each of the two oscillators, we know that

$$\left(\frac{dP}{d\Omega}\right)_{p_x}\;\propto\;\sin^2(\text{angle between x axis and }\mathbf{n})\;=\;1\,-\,(\mathbf{n}\cdot\hat{\mathbf{x}})^2\;=\;1\,-\,\sin^2\theta\cos^2\phi, \qquad\text{(S.41)}$$

and likewise

$$\left(\frac{dP}{d\Omega}\right)_{p_y}\;\propto\;\sin^2(\text{angle between y axis and }\mathbf{n})\;=\;1\,-\,(\mathbf{n}\cdot\hat{\mathbf{y}})^2\;=\;1\,-\,\sin^2\theta\sin^2\phi. \qquad\text{(S.42)}$$

Adding up these two angular distributions, we get

$$\begin{aligned}\left(\frac{dP}{d\Omega}\right)_{\rm net} &\propto\ 1\ -\ \sin^2\theta\cos^2\phi\ +\ 1\ -\ \sin^2\theta\sin^2\phi\\ &=\ 2\ -\ \sin^2\theta(\cos^2\phi+\sin^2\phi\ =\ 1)\\ &=\ 1\ +\ \cos^2\phi,\end{aligned}\tag{S.43}$$

precisely as we got in eq. (S.39).

(c) Integrating the power per solid angle (S.39) over the directions **n**, we get

$$\begin{aligned}P_{\rm net} &= \oiint d\Omega_{\mathbf{n}}\,\frac{dP}{d\Omega}\\ &= \frac{p_0^2\omega^4\mu_0}{16\pi^2c}\oiint\frac{1+\cos^2\theta}{2}\,d\Omega_{\mathbf{n}}\\ &= \frac{p_0^2\omega^4\mu_0}{16\pi^2c}\times\frac{8\pi}{3}\\ &= \frac{p_0^2\omega^4\mu_0}{6\pi c}\,,\end{aligned}\tag{S.44}$$

or in terms of the λ_0 and b parameters of the ring,

$$P_{\rm net}\ =\ \frac{\pi^2\lambda_0^2b^4\omega^4\mu_0}{6\pi c}\,.\tag{S.45}$$

Problem **4**:

(a) The radiation intensity $I\ =\ |\,\langle\mathbf{S}\rangle_{\rm rad}\,|$ is inversely proportional to the square of the distance from the antenna. It also depends on the direction: for the magnetic dipole antenna, $I\propto\sin^2\alpha$ where α is the angle between the dipole axis and the direction to the observer. For the vertical antenna, we may identify α with the θ angle in the spherical coordinate system, thus

$$I\ =\ \text{const}\times\frac{\sin^2\theta}{r^2}\,.\tag{S.46}$$

For an observer on the ground at distance ℓ from the bottom of the tower while the antenna is at the top of the tower of height h, we have

$$r^2 \;=\; \ell^2 \;+\; h^2 \tag{S.47}$$

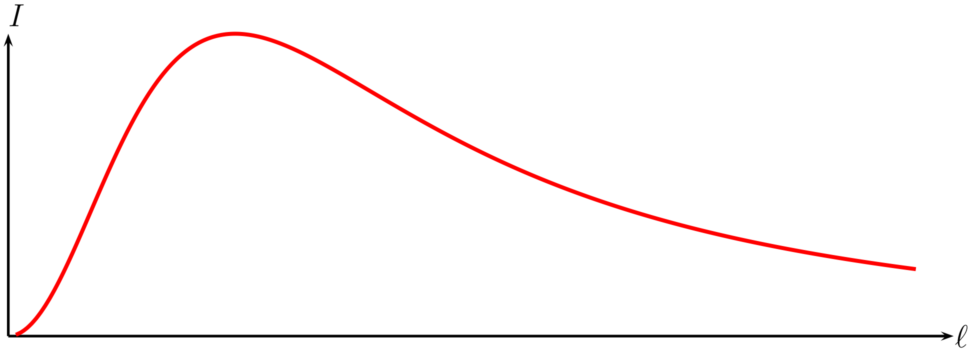
while

$$\sin^2\theta \;=\; \frac{\ell^2}{r^2} \;=\; \frac{\ell^2}{\ell^2+h^2}\,, \tag{S.48}$$

thus

$$I \;=\; \text{const}\times\frac{\ell^2}{(\ell^2+h^2)^2}\,. \tag{S.49}$$

(b) Let's plot the intensity I as a function of the distance from the tower base:



We see that there is little radiation directly below the antenna, but as we move away, the intensity increases, reaches the maximum and then starts decreasing. The maximum lies when

$$\frac{dI}{d\ell} \;=\; \text{const}\times\frac{2\ell(h^2-\ell^2)}{(h^2+\ell^2)^3} \;=\; 0, \tag{S.50}$$

which happens for $\ell = h$.

Now, where should the inspector check the radiation intensity to make sure it complies with the city regulations? In principle, one should check everywhere withing the city limits, but that's impractical. Instead, once we know where the intensity should be maximal, it would be enough to measure the intensity at that point, $\ell = h = 200$ m away from the tower's base. If the radiation is in compliance at that point, it should be in compliance everywhere, so the inspector can just as well save the extra work of measuring elsewhere. But if the radiation at that point exceed the regulation, then the station is out of compliance, period. So a competent and honest inspector should check the radiation intensity 200 m from the tower base.

Instead, the city engineer went to check the radiation intensity at the tower base — the point where there should be very little radiation in any case — and did not bother checking it anywhere else. This could have been cased by incompetence (perhaps aggravated by too much work and too little pay), or it could have been done on purpose to find the radio station in compliance even if it is not. Alas, we do not have enough data to decide between the incompetence and the dishonesty...

(c) The directional intensity of the magnetic dipole radiation is

$$I = \frac{\sin^2\theta}{r^2} \times \frac{\mu_0\omega^4\tilde{m}^2}{32\pi^2c^3} \tag{S.51}$$

where $\tilde{m}$ is the amplitude of the antenna's magnetic dipole moment. The net power radiated by the same antenna is

$$P_{\text{net}} = \frac{\mu_0\omega^4\tilde{m}^2}{12\pi c^3}\,, \tag{S.52}$$

so we may express the directional intensity of the antenna in terms of its net power as

$$I = \frac{3}{8\pi}P_{\text{net}} \times \frac{\sin^2\theta}{r^2} \tag{S.53}$$

regardless of any of the antenna's details. For the intensity (S.49) measure at the ground level, this means

$$I = \frac{3}{8\pi}P_{\text{net}} \times \frac{\ell^2}{(\ell^2+h^2)^2}\,. \tag{S.54}$$

In particular, the maximal intensity at the ground level is

$$I_{\max} = \frac{3}{8\pi}P_{\text{net}} \times \frac{1}{4h^2} = \frac{3P_{\text{net}}}{32\pi h^2}\,. \tag{S.55}$$

Thus, for a station of net broadcast power $P_{\text{net}} = 35$ kW whose antenna is 200 m above the ground, the maximal radio intensity on the ground is

$$I_{\rm max} \;=\; \frac{3}{32\pi}\,\frac{35,000\ {\rm W}}{(200\ {\rm m})^2} \;=\; 26\ {\rm mW/m}^2. \tag{S.56}$$

This is a bit higher than the city limit of 20 $\rm mW/m^2$, so the KRUD station is out of compliance.

<u>Problem **5**:</u>

(a) The dipole magnetic field at the surface of the star is

$$\mathbf{B} \;=\; \frac{\mu_0}{4\pi R^3}\bigl(\mathbf{m} \;-\; 3\mathbf{n}(\mathbf{n}\cdot\mathbf{m})\bigr), \tag{S.57}$$

so its magnitude varies between $\mu_0 m/4\pi R^3$ at the magnetic equator and $2\times\mu_0 m/4\pi R^3$ at the magnetic poles. Taking the $B_{\rm max} = 6\cdot 10^8$ T as the highest value at the magnetic poles, we get

$$m \;=\; \frac{2\pi R^3}{\mu_0}\times B_{\rm max} \;=\; 3\cdot 10^{27}\ {\rm A\,m}^2. \tag{S.58}$$

(b) The z component (along the star's rotation axis) of the magnetic dipole moment (3) is constant, while the x and the y components oscillate with similar real amplitudes $m_0\sin\alpha$ and phases differing by 90°. In terms of the complex amplitudes,

$$\mathbf{m}(t) \;=\; {\rm Re}\bigl[\tilde{\mathbf{m}}e^{-i\omega t}\bigr] \;+\; (t\text{-independent constant}) \tag{S.59}$$

for

$$\tilde{\mathbf{m}} \;=\; m_0\sin\alpha(\hat{\mathbf{x}}+i\hat{\mathbf{y}}). \tag{S.60}$$

As explained in class, the EM radiation by this magnetic dipole moment has directional power

$$\frac{dP}{d\Omega} = \frac{Z_0\omega^4}{32\pi^2c^4}\,\|\mathbf{n}\times\tilde{\mathbf{m}}\|^2 \tag{S.61}$$

where

$$\|\mathbf{n}\times\tilde{\mathbf{m}}\|^2 = (\mathbf{n}\times\tilde{\mathbf{m}})\cdot(\mathbf{n}\times\tilde{\mathbf{m}}^*) = \mathbf{m}\cdot\tilde{\mathbf{m}}^* - |\mathbf{n}\cdot\mathbf{m}|^2. \tag{S.62}$$

Specifically, for the dipole momentum amplitude (S.60),

$$\|\mathbf{n}\times\tilde{\mathbf{m}}\|^2 = m_0^2\sin^2\alpha\big(\|\hat{\mathbf{x}}+i\hat{\mathbf{y}}\|^2 - |\mathbf{n}\cdot(\hat{\mathbf{x}}+i\hat{\mathbf{y}})|^2\big), \tag{S.63}$$

we have

$$\frac{dP}{d\Omega} = \frac{Z_0\omega^4m_0^2\sin^2\alpha}{32\pi^2c^4}\big(\|\hat{\mathbf{x}}+i\hat{\mathbf{y}}\|^2 - |\mathbf{n}\cdot(\hat{\mathbf{x}}+i\hat{\mathbf{y}})|^2\big). \tag{S.64}$$

(c) The angular distribution of the 30 Hz radiation follows from the last factor of eq. (S.64):

$$\big(\|\hat{\mathbf{x}}+i\hat{\mathbf{y}}\|^2 - |\mathbf{n}\cdot(\hat{\mathbf{x}}+i\hat{\mathbf{y}})|^2\big) = 2 - \left|\sin\theta e^{i\phi}\right|^2 = 2 - \sin^2\theta = 1 + \cos^2\theta. \tag{S.65}$$

Thus, the Crab radiates in all directions, but the power radiated along its rotation axis is twice as strong as the power radiated ⊥ to the rotation axis.

(d) Combining results from eqs. (S.64) and (S.65), we have directional power

$$\frac{dP}{d\Omega} = \frac{Z_0\omega^4m_0^2\sin^2\alpha}{32\pi^2c^4}\times(1+\cos^2\theta). \tag{S.66}$$

Integrating this directional power over 4π directions of $\mathbf{n}$, we have

$$\oiint(1+\cos^2\theta)\,d\Omega = 4\pi + \frac{4\pi}{3} = \frac{16\pi}{3} \tag{S.67}$$

and hence net EM power

$$P_{\text{net}} = \frac{Z_0\omega^4m_0^2\sin^2\alpha}{6\pi c^4}. \tag{S.68}$$

In this formula, m_0 is the net magnetic dipole moment of the Crab pulsar; in part (a) we have estimated $m_0 \approx 3\cdot10^{27}$ A m^2. Also, α is the angle between the rotational and the magnetic axes of the pulsar; observationally, $\theta \approx 45^\circ$ so $\sin^2\theta \approx \frac{1}{2}$. The rest of the parameters in eq. (S.68) are constants of Nature; plugging them in leads us to $P_{\text{net}} = 14\cdot10^{30}$ W.

Problem **6**:

(a) The quadrupole moment tensor of a system of point charges is

$$Q_{ij} = \sum_n q_n \left(\tfrac{3}{2} r_{n,i} r_{n,j} - \tfrac{1}{2}\mathbf{r}_n^2\right). \tag{S.69}$$

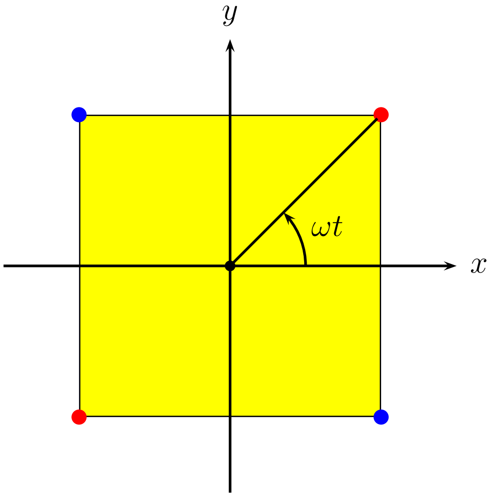
The 4 charges in question are all in the same plane — which we take to be the (x, y) plane, — hence $Q_{xz} = Q_{yz} = 0$. Also, all 4 charges lie at the same distance $r = a/\sqrt{2}$ from the origin and the net charge $\sum_n q_n$ vanishes, hence $\sum_n q_n r_n^2 = 0$ and therefore

$$Q_{zz} = 0 \quad \text{and} \quad Q_{xx} + Q_{yy} = 0. \tag{S.70}$$

The remaining independent components of the quadrupole tensor form a complex combination

$$\mathcal{Q} = Q_{xx} - Q_{yy} + 2iQ_{xy} = \frac{3}{2}\sum_n q_n (x_n + iy_n)^2. \tag{S.71}$$

For the charges at the corners of a rotating square



$$\begin{aligned} \text{charge } q_n &= (-1)^n q \\ \text{at } x_n + iy_n &= i^n \times e^{i\omega t} \times \frac{a}{\sqrt{2}} \\ \text{for } n &= 0, 1, 2, 3. \end{aligned} \tag{S.72}$$

we have

$$\forall n: \quad q_n(x_n + iy_n)^2 = +\frac{qa^2}{2} \times e^{2i\omega t} \tag{S.73}$$

and hence

$$\mathcal{Q} = 3qa^2 \times e^{2i\omega t}. \tag{S.74}$$

In terms of the quadrupole tensor components, this means

$$Q_{xx} = -Q_{yy} = \tfrac{1}{2}\,\mathrm{Re}(\mathcal{Q}) = \tfrac{3}{2}qa^2 \times \cos(2\omega t), \qquad Q_{xy} = \tfrac{1}{2}\,\mathrm{Im}(\mathcal{Q}) = \tfrac{3}{2}qa^2 \times \sin(2\omega t), \tag{S.75}$$

or in matrix notations

$$\overleftrightarrow{Q}(t) = \frac{3qa^2}{2}\begin{pmatrix} +\cos(2\omega t) & +\sin(2\omega t) & 0 \\ +\sin(2\omega t) & -\cos(2\omega t) & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{S.76}$$

Note that this quadrupole tensor oscillates with frequency 2ω, twice the rotation frequency of the square.

As to the complex amplitude of the quadrupole oscillation,

$$\overleftrightarrow{Q}(t) = \frac{3qa^2}{2}\,\mathrm{Re}\left[e^{-2i\omega t}\begin{pmatrix} +1 & +i & 0 \\ +i & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right], \tag{S.77}$$

hence amplitude

$$\widetilde{\overleftrightarrow{Q}} = \frac{3qa^2}{2}\begin{pmatrix} +1 & +i & 0 \\ +i & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{S.78}$$

(b–c) As explained in class, the EM power radiated in a particular direction $\mathbf{n}$ is

$$\frac{dP}{d\Omega} = \frac{Z_0\omega_{\mathrm{osc}}^2}{2c^2} \times \left(|\mathbf{f}(\mathbf{n})|^2 - |\mathbf{n}\cdot\mathbf{f}(\mathbf{n})|^2\right) \tag{S.79}$$

where

$$\mathbf{f}(\mathbf{n}) = \frac{1}{4\pi}\iiint d^3\mathbf{r}'\,\mathbf{J}(\mathbf{r}')\,\exp(-ik\mathbf{n}\cdot\mathbf{r}'). \tag{S.80}$$

In the long wavelength approximation, the leading contribution to the $\mathbf{f}$ comes from the lowest oscillating multipole moment, electric or magnetic. For the system at hand, the lowest oscillating moment is the electric quadrupole; as we saw in part (a), it has frequency $\omega_{\rm osc} = 2\omega$ and amplitude (S.78). For a general electric quadrupole,

$$f_j(\mathbf{n}) \;=\; \frac{\omega_{\rm osc}^2}{12\pi c}\,\widetilde{Q}_{jk} n_k\,, \tag{S.81}$$

so for the quadrupole in question

$$\begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix} \;=\; \frac{\omega_{\rm osc}^2 q a^2}{8\pi c} \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} (n_x + i n_y), \tag{S.82}$$

or in spherical coordinates

$$\begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix} \;=\; \frac{\omega_{\rm osc}^2 q a^2}{8\pi c} \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} \sin\theta\, e^{i\phi}. \tag{S.83}$$

Consequently,

$$\mathbf{f}^* \cdot \mathbf{f} \;=\; \frac{\omega_{\rm osc}^4 q^2 a^4}{64\pi^2 c^2} \times 2\sin^2\theta, \tag{S.84}$$

$$\mathbf{n} \cdot \mathbf{f} \;=\; \frac{\omega_{\rm osc}^4 q^2 a^4}{64\pi^2 c^2} \times \left(\sin\theta\, e^{i\phi}\right)^2, \tag{S.85}$$

hence

$$\Big(|\mathbf{f}(\mathbf{n})|^2 \,-\, |\mathbf{n}\cdot\mathbf{f}(\mathbf{n})|^2\Big) \;=\; \frac{\omega_{\rm osc}^4 q^2 a^4}{64\pi^2 c^2} \times \Big(2\sin^2\theta \,-\, \sin^4\theta\Big), \tag{S.86}$$

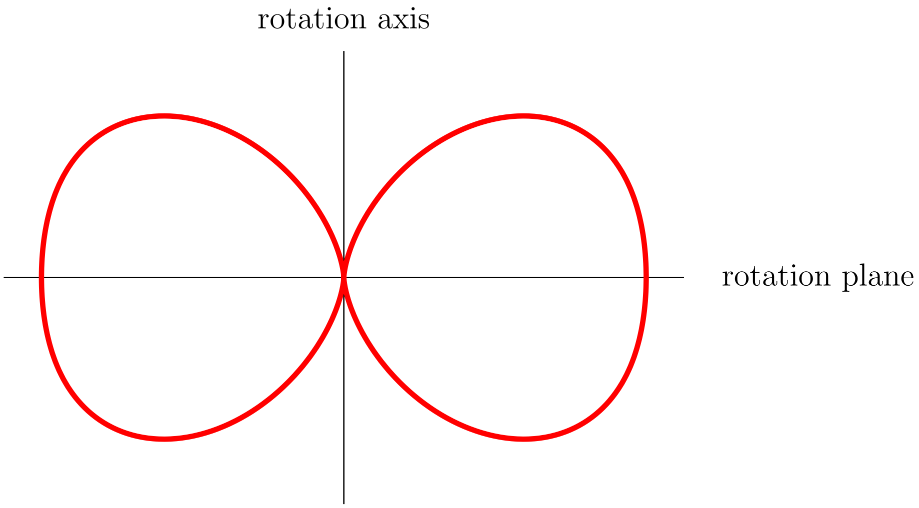
and therefore

$$\frac{dP}{d\Omega} \;=\; \frac{Z_0 q^2 a^4 \omega_{\rm osc}^6}{128\pi^2 c^4} \times \sin^2\theta(2 - \sin^2\theta). \tag{S.87}$$

In particular, the angular dependence of the radiated power has form

$$\frac{dP}{d\Omega} \;\propto\; \sin^2\theta(2 - \sin^2\theta) \;=\; 1 \,-\, \cos^4\theta. \tag{S.88}$$

Graphically,



As to the total power radiated by the rotating quadrupole,

$$P_{\text{net}} \;=\; \frac{Z_0 q^2 a^4 \omega_{\text{osc}}^6}{128\pi^2 c^4} \times \oiint d^2\Omega\,(1-\cos^4\theta) \tag{S.89}$$

where $\omega_{\text{osc}} = 2\omega$ and

$$\oiint d^2\Omega\,(1-\cos^4\theta) \;=\; 2\pi\int\limits_{-1}^{+1} d\cos\theta\,(1-\cos^4\theta) \;=\; 4\pi\times\left(1-\frac{1}{5}\right) \;=\; \frac{16\pi}{5}\,. \tag{S.90}$$

Thus altogether,

$$P_{\text{net}} \;=\; \frac{8Z_0 q^2 q^4 \omega^6}{5\pi c^4} \;=\; \frac{8q^2}{5\pi\epsilon_0}\times\frac{a^4\omega^6}{c^5}\,. \tag{S.91}$$

Problem **8**, for extra credit:

(a) The Poynting vector $\mathbf{S}=\mathbf{E}\times\mathbf{H}$ is the density of the energy flow rate. So if we surround the retarded position of the accelerating particle with a large sphere of radius R, then the rate of energy flowing through the $d\Omega$ sector of the sphere (per unit of the observer time) is

$$\frac{dU}{dt\times d\Omega} \;=\; R^2\mathbf{n}\cdot\mathbf{S}(\mathbf{r}=R\mathbf{n}) \tag{S.92}$$

To change that to the rate of energy leaving the particle per unit of the particle's retarded time, we multiply by

$$\frac{dt_{\text{obs}}}{dt_{\text{ret}}} \;=\; (1-\boldsymbol{\beta}\cdot\mathbf{n}), \tag{S.93}$$

then take the $R\to\infty$ limit to select only the energy that is radiated all the way to infinity, and integrate over the direction of $\mathbf{n}$, thus

$$\frac{dU_{\text{em}}}{dt_{\text{particle}}} \;=\oiint d\Omega_{\mathbf{n}}\,(1-\boldsymbol{\beta}\cdot\mathbf{n})\lim_{R\to\infty}\Big(R^2\mathbf{n}\cdot\mathbf{S}(R\mathbf{n})\Big). \tag{S.94}$$

The net EM momentum radiated by the particle works in the same way, except for replacing the Poynting vector — which is the density of the of energy flow rate — with the (minus) stress tensor $-\overleftrightarrow{T}$, which is the density of the momentum flow rate. In components, $-T^{ij}$ is the density in the j direction of the flow rate of the i^{th} component of the momentum, so the rate of the flow though sector $d\Omega$ of the distant sphere is

$$-T^{ij}d\text{area}^j \;=\; -T^{ij}(\mathbf{r}=R\mathbf{n})R^2n^j\,d\Omega \tag{S.95}$$

(for the i^{th} component of the momentum), or in vector notations

$$\Big(-R^2\,\overleftrightarrow{T}(R\mathbf{n})\cdot\mathbf{n}\Big)\,d\Omega \tag{S.96}$$

Consequently, the momentum analogue of eq. (S.94) for the energy is

$$\frac{d\mathbf{P}_{\text{em}}}{dt_{\text{particle}}} \;=\oiint d\Omega_{\mathbf{n}}\,(1-\boldsymbol{\beta}\cdot\mathbf{n})\lim_{R\to\infty}\Big(-R^2\,\overleftrightarrow{T}(R\mathbf{n})\cdot\mathbf{n}\Big). \tag{4}$$

Finally, the angular momentum density follows from the linear momentum density as

$$\vec{\ell}(\mathbf{r}) \;=\; \mathbf{r}\times\mathbf{g}(\mathbf{r}), \tag{S.97}$$

and the same relation applies to the densities of rates at which these momenta flow,

$$\text{density}^j[\text{rate of }\mathbf{L}\text{ flow}] \;=\; \mathbf{r}\times\text{density}^j[\text{rate of }\mathbf{P}\text{ flow}]. \tag{S.98}$$

Consequently, the rate of the angular momentum flowing through the sector $d\Omega$ of the distant sphere is

$$\mathbf{r}\times\left(-R^2\overset{\leftrightarrow}{T}(R\mathbf{n})\cdot\mathbf{n}\right)d\Omega \;=\; \Big(-R^3\mathbf{n}\times\big(\overset{\leftrightarrow}{T}(R\mathbf{n})\cdot\mathbf{n}\big)\Big)\,d\omega, \tag{S.99}$$

and therefore — in complete analogy to eqs. (S.94) for the energy and (4) for the linear momentum, — the rate of EM angular momentum radiated by the particle is

$$\frac{d\mathbf{L}_{\text{em}}}{dt_{\text{particle}}} \;=\oiint d\Omega_{\mathbf{n}}\,(1-\boldsymbol{\beta}\cdot\mathbf{n})\lim_{R\to\infty}\Big(-R^3\mathbf{n}\big(\overset{\leftrightarrow}{T}(R\mathbf{n})\cdot\mathbf{n}\big)\Big). \tag{5}$$

(b) The EM stress tensor is

$$\begin{aligned} T^{ij} \;&=\; \epsilon_0E^iE^j \,+\, \frac{1}{\mu_0}B^iB^j \,-\, \frac{\delta^{ij}}{2}\left(\epsilon_0\mathbf{E}^2 \,+\, \frac{1}{\mu_0}\mathbf{B}^2\right) \\ &=\; \epsilon_0\left(E^iE^j \,+\, c^2B^iB^j \,-\, \frac{\delta^{ij}}{2}\Big(\mathbf{E}^2 \,+\, c^2\mathbf{B}^2\Big)\right). \end{aligned} \tag{S.100}$$

Contracting the j index of this tensor with that of the unit vector n^j, we have

$$\begin{aligned} \left(\overset{\leftrightarrow}{T}\cdot\mathbf{n}\right)^i \;=\; T^{ij}n^j \;&=\; \epsilon_0\left(E^iE^jn^j \,+\, c^2B^iB^jn^j \,-\, \frac{\delta^{ij}n^j}{2}\Big(\mathbf{E}^2 \,+\, c^2\mathbf{B}^2\Big)\right) \\ &=\; \epsilon_0\left(\mathbf{E}(\mathbf{E}\cdot\mathbf{n}) \,+\, c^2\mathbf{B}(\mathbf{B}\cdot\mathbf{n}) \,-\, \frac{\mathbf{n}}{2}\Big(\mathbf{E}^2 \,+\, c^2\mathbf{B}^2\Big)\right). \end{aligned} \tag{S.101}$$

Next, for $c\mathbf{B}=\mathbf{n}\times\mathbf{E}$ we have $(\mathbf{B}\cdot\mathbf{n})=0$ while

$$c^2\mathbf{B}^2 \;=\; (\mathbf{n}\times\mathbf{E})^2 \;=\; \mathbf{E}^2 \,-\, (\mathbf{n}\cdot\mathbf{E})^2 \tag{S.102}$$

and hence

$$\tfrac{1}{2}\Bigl(\mathbf{E}^2\,+\,c^2\mathbf{B}^2\Bigr)\;=\;\mathbf{E}^2\,-\,\tfrac{1}{2}(\mathbf{n}\cdot\mathbf{E})^2. \tag{S.103}$$

Altogether, this gives us

$$\overset{\leftrightarrow}{T}\cdot\mathbf{n}\;=\;\epsilon_0\Bigl(\mathbf{E}(\mathbf{n}\cdot\mathbf{E})\,-\,\mathbf{n}\Bigl(\mathbf{E}^2\,-\,\tfrac{1}{2}(\mathbf{n}\cdot\mathbf{E})^2\Bigr)\Bigr) \tag{S.104}$$

and hence eq. (6).

(c) The electric field of a moving charge is a sum of the generalized Coulomb term and the radiation term,

$$\mathbf{E}\;=\;\mathbf{E}_{\rm g.C.}\,+\,\mathbf{E}_{\rm rad} \tag{S.105}$$

where $\mathbf{E}_{\rm rad}\perp\mathbf{n}$. Consequently, in eq. (6) we end up with

$$-\overset{\leftrightarrow}{T}\cdot\mathbf{n}\;=\;\epsilon_0\Bigl(\mathbf{n}\Bigl(\mathbf{E}^2_{\rm g.C.}+2\mathbf{E}_{\rm g.C.}\cdot\mathbf{E}_{\rm rad}+\mathbf{E}^2_{\rm rad}-(\mathbf{n}\cdot\mathbf{E}_{g.C.})^2\Bigr)-(\mathbf{E}_{g.C.}+\mathbf{E}_{\rm rad})(\mathbf{n}\cdot\mathbf{E}_{g.C.})\Bigr). \tag{S.106}$$

Moreover, in the $R\to\infty$ limit, the $\mathbf{E}_{\rm rad}$ scales as $1/R$ while the $\mathbf{E}_{g.C.}$ scales as $1/R^2$, so only the terms containing 2 factors of $\mathbf{E}_{\rm rad}$ and none of the $\mathbf{E}_{g.C.}$ survive in the $R\to\infty$ limit of

$$\lim_{R\to\infty}\Bigl(-R^2\,\overset{\leftrightarrow}{T}(R\mathbf{n})\cdot\mathbf{n}\Bigr)\;=\;\epsilon_0\mathbf{n}\Bigl(R^2\mathbf{r}^2_{\rm rad}\Bigr)^2. \tag{S.107}$$

Consequently, eq. (4) becomes

$$\begin{aligned}\frac{d\mathbf{P}_{\rm em}}{dt_{\rm particle}} &= \oiint d\Omega_{\mathbf{n}}\,(1-\boldsymbol{\beta}\cdot\mathbf{n})\epsilon_0\mathbf{n}\Bigl(R^2\mathbf{E}^2_{\rm rad}\Bigr)\\ &= \oiint d\Omega_{\mathbf{n}}\,(1-\boldsymbol{\beta}\cdot\mathbf{n})\epsilon_0\mathbf{n}\,\frac{Q^2}{4\pi\epsilon_0^2c^2}\,\frac{[\mathbf{n}\times((\mathbf{n}-\boldsymbol{\beta})\times\mathbf{a})]^2}{(1-\boldsymbol{\beta}\cdot\mathbf{n})^6}\\ &= \frac{Q^2}{16\pi^2\epsilon_0c^4}\oiint d\Omega_{\mathbf{n}}\,\mathbf{n}*\frac{[\mathbf{n}\times((\mathbf{n}-\boldsymbol{\beta})\times\mathbf{a})]^2}{(1-\boldsymbol{\beta}\cdot\mathbf{n})^5}\,.\end{aligned} \tag{S.108}$$

Next, let's expand the bottom line here in powers of $\boldsymbol{\beta}=\mathbf{v}/c$ and keep the first power of $\boldsymbol{\beta}$

but not the higher powers. In the numerator we have

$$\begin{aligned}[\mathbf{n}\times((\mathbf{n}-\boldsymbol{\beta})\times\mathbf{a})]^2\ &\approx\ [\mathbf{n}\times(\mathbf{n}\times\mathbf{a})]^2\ -\ 2[\mathbf{n}\times(\mathbf{n}\times\mathbf{a})]\cdot[\mathbf{n}\times(\boldsymbol{\beta}\times\mathbf{a})]\\ &=\ (\mathbf{n}\times\mathbf{a})^2\ -\ 2(\mathbf{n}\times\mathbf{a})\cdot(\boldsymbol{\beta}\times\mathbf{a})\\ &=\ \mathbf{a}^2\ -\ (\mathbf{n}\cdot\mathbf{a})^2\ -\ 2(\mathbf{n}\cdot\boldsymbol{\beta})\mathbf{a}^2\ +\ 2(\mathbf{n}\cdot\mathbf{a})(\boldsymbol{\beta}\cdot\mathbf{a}),\end{aligned}\tag{S.109}$$

while in the denominator

$$\frac{1}{(1-\boldsymbol{\beta}\cdot\mathbf{n})^5}\ \approx\ 1\ +\ 5(\boldsymbol{\beta}\cdot\mathbf{n}),\tag{S.110}$$

so altogether

$$\begin{aligned}\frac{[\mathbf{n}\times((\mathbf{n}-\boldsymbol{\beta})\times\mathbf{a})]^2}{(1-\boldsymbol{\beta}\cdot\mathbf{n})^5}\ \approx\ &\big(\mathbf{a}^2\ -\ (\mathbf{n}\cdot\mathbf{a})^2\big)\ +\ 5(\boldsymbol{\beta}\cdot\mathbf{n})\big(\mathbf{a}^2\ -\ (\mathbf{n}\cdot\mathbf{a})^2\big)\\ &-2(\boldsymbol{\beta}\cdot\mathbf{n})\mathbf{a}^2\ +\ 2(\boldsymbol{\beta}\cdot\mathbf{a})(\mathbf{n}\cdot\mathbf{a}).\end{aligned}\tag{S.111}$$

Finally, we integrate over the direction of $\mathbf{n}$,

$$\oiint d\Omega_{\mathbf{n}}\Big(\cdots\Big)\mathbf{n}.\tag{S.112}$$

By $\mathbf{n}\to-\mathbf{n}$ symmetry of the integral, only the even terms contribute to the integral, so in light of the overall factor of $\mathbf{n}$, the terms inside $(\cdots)$ must be odd WRT $\mathbf{n}\to-\mathbf{n}$. A quick look at eq. (S.111) tells us that the first term is even while the rest of the terms are odd, thus (after a bit of algebra)

$$\oiint d\Omega_{\mathbf{n}}\Big(\cdots\Big)\mathbf{n}\ =\oiint d\Omega_{\mathbf{n}}\Big(3(\mathbf{n}\cdot\boldsymbol{\beta})\mathbf{a}^2\ -\ 5(\mathbf{n}\cdot\boldsymbol{\beta})(\mathbf{n}\cdot\mathbf{a})^2\ +\ 2(\mathbf{n}\cdot\mathbf{a})(\boldsymbol{\beta}\cdot\mathbf{a})\Big)\mathbf{n}.\tag{S.113}$$

At this point, we use

$$\oiint d\Omega_{\mathbf{n}}\,n^in^j\ =\ \frac{4\pi}{3}\,\delta^{ij}\,,\tag{S.114}$$

$$\oiint d\Omega_{\mathbf{n}}\,n^in^jn^kn^\ell\ =\ \frac{4\pi}{15}\big(\delta^{ij}\delta^{k\ell}+\delta^{ik}\delta^{j\ell}+\delta^{i\ell}\delta^{jk}\big),\tag{S.115}$$

to integrate

$$\oiint d\Omega_{\mathbf{n}}\,(\mathbf{n}\cdot\boldsymbol{\beta})\mathbf{n}\ =\ \frac{4\pi}{3}\,\boldsymbol{\beta},\tag{S.116}$$

$$\oiint d\Omega_{\mathbf{n}}\,(\mathbf{n}\cdot\mathbf{a})\mathbf{n} = \frac{4\pi}{3}\,\mathbf{a},\text{(S.117)}$$

$$\oiint d\Omega_{\mathbf{n}}\,(\mathbf{n}\cdot\boldsymbol{\beta})(\mathbf{n}\cdot\mathbf{a})^2\mathbf{n} \;=\; \frac{4\pi}{15}\big(\boldsymbol{\beta}\mathbf{a}^2 \,+\, 2\mathbf{a}(\boldsymbol{\beta}\cdot\mathbf{a}), \tag{S.118}$$

hence altogether

$$\begin{aligned} &\oiint d\Omega_{\mathbf{n}}\Big(3(\mathbf{n}\cdot\boldsymbol{\beta})\mathbf{a}^2 \,-\, 5(\mathbf{n}\cdot\boldsymbol{\beta})(\mathbf{n}\cdot\mathbf{a})^2 \,+\, 2(\mathbf{n}\cdot\mathbf{a})(\boldsymbol{\beta}\cdot\mathbf{a})\Big)\mathbf{n} \\ &\qquad =\; 3\mathbf{a}^2 * \frac{4\pi}{3}\,\boldsymbol{\beta} \;-\; 5*\frac{4\pi}{15}\big(\boldsymbol{\beta}\mathbf{a}^2 \,+\, 2\mathbf{a}(\boldsymbol{\beta}\cdot\mathbf{a}) \;+\; 2(\boldsymbol{\beta}\cdot\mathbf{a})*\frac{4\pi}{3}\,\mathbf{a} \\ &\qquad =\; \frac{8\pi}{3}\,\mathbf{a}^2\boldsymbol{\beta}. \end{aligned} \tag{S.119}$$

Finally, plugging this formula into eq. (S.108), we obtain

$$\frac{d\mathbf{P}_{\text{em}}}{dt_{\text{particle}}} \;=\; \frac{Q^2}{16\pi^2\epsilon_0 c^4} * \frac{8\pi}{3}\,\mathbf{a}^2\boldsymbol{\beta} \;=\; \frac{Q^2\mu_0}{6\pi c^2} * \mathbf{a}^2\boldsymbol{\beta} \;=\; \frac{Q^2\mu_0}{6\pi c^3} * \mathbf{a}^2\mathbf{v}, \tag{8}$$

Quod erat demonstrandum.

(d) For the stress tensor — or rather the $-\overset{\leftrightarrow}{T}\cdot\mathbf{n}$ as in eq. (6), we have

$$\mathbf{n}\times\big(-\overset{\leftrightarrow}{T}\cdot\mathbf{n}\big) \;=\; -\epsilon_0(\mathbf{n}\times\mathbf{E})(\mathbf{n}\cdot\mathbf{E}). \tag{S.120}$$

In terms of $\mathbf{E} = \mathbf{E}_{\text{g.C.}} + \mathbf{E}_{\text{rad}}$, only the Coulomb field contributes to the $(\mathbf{n}\cdot\mathbf{E})$ because $\mathbf{E}_{\text{rad}} \perp \mathbf{n}$. Consequently,

$$\mathbf{n}\times\big(-\overset{\leftrightarrow}{T}\cdot\mathbf{n}\big) \;=\; -\epsilon_0(\mathbf{n}\times\mathbf{E}_{\text{rad}})(\mathbf{n}\cdot\mathbf{E}_{g.C.}) \;-\; \epsilon_0(\mathbf{n}\times\mathbf{E}_{g.C.})(\mathbf{n}\cdot\mathbf{E}_{g.C.}), \tag{S.121}$$

where the first term scales with the radius R as $1/R^3$ while the second terms scales as R^4. In the context of eq. (5) for the angular momentum radiation we want only the leading $1/R^3$

term, thus

$$\begin{aligned}\lim_{R\to\infty}\Big(-R^3\mathbf{n}\times\big(\overset{\leftrightarrow}{T}(R\mathbf{n}\cdot\mathbf{n}\big) \;&=\; -\epsilon_0(R\mathbf{n}\times\mathbf{E}_{\rm rad})(R^2\mathbf{n}\cdot\mathbf{E}_{g.C.})\\ &=\; -\frac{Q^2}{16\pi^2\epsilon_0c^2}*\frac{\mathbf{n}\times(\mathbf{n}\times((\mathbf{n}-\boldsymbol{\beta})\times\mathbf{a}))}{(1-\boldsymbol{\beta}\cdot\mathbf{n})^3}*\frac{(1-\beta^2)(1-\mathbf{n}\cdot\boldsymbol{\beta})}{(1-\boldsymbol{\beta}\cdot\mathbf{n})^3}\end{aligned} \tag{S.122}$$

and therefore

$$\frac{d\mathbf{L}_{\rm em}}{dt_{\rm particle}} \;=\; -\frac{Q^2\mu_0}{16\pi^2}\oiint d\Omega_{\mathbf{n}}\,(1-\beta^2)\frac{\mathbf{n}\times(\mathbf{n}\times((\mathbf{n}-\boldsymbol{\beta})\times\mathbf{a}))}{(1-\boldsymbol{\beta}\cdot\mathbf{n})^4}\,. \tag{S.123}$$

Now let's expand the integrand here to the first order in $\boldsymbol{\beta}$. In the numerator

$$(1-\beta^2)\;\approx\;1 \tag{S.124}$$

and

$$\begin{aligned}\mathbf{n}\times(\mathbf{n}\times((\mathbf{n}-\boldsymbol{\beta})\times\mathbf{a})) \;&=\; \mathbf{n}\times(\mathbf{n}\times(\mathbf{n}\times\mathbf{a}))\;-\;\mathbf{n}\times(\mathbf{n}\times(\boldsymbol{\beta}\times\mathbf{a}))\\ &=\; -\mathbf{n}\times\mathbf{a}\;+\;\boldsymbol{\beta}\times\mathbf{a}\;-\mathbf{n}(\mathbf{n}\cdot(\boldsymbol{\beta}\times\mathbf{a})),\end{aligned} \tag{S.125}$$

while in the denominator

$$\frac{1}{(1-\boldsymbol{\beta}\cdot\mathbf{n})^4}\;\approx\;1\;+\;4(\boldsymbol{\beta}\cdot\mathbf{n}), \tag{S.126}$$

thus altogether

$$\begin{aligned}(1-\beta^2)\frac{\mathbf{n}\times(\mathbf{n}\times((\mathbf{n}-\boldsymbol{\beta})\times\mathbf{a}))}{(1-\boldsymbol{\beta}\cdot\mathbf{n})^4}\;&\approx\\ \approx\;-\mathbf{n}\times\mathbf{a}\;-\;4(\boldsymbol{\beta}\cdot\mathbf{n})(\mathbf{n}\times\mathbf{a})\;&+\;\boldsymbol{\beta}\times\mathbf{a}\;-\mathbf{n}(\mathbf{n}\cdot(\boldsymbol{\beta}\times\mathbf{a})).\end{aligned} \tag{S.127}$$

Integrating this expression over the directions $\mathbf{n}$, we get

$$\oiint d\Omega_{\mathbf{n}}\,(-\mathbf{n}\times\mathbf{a})\;=\;0, \text{(S.128)}$$

$$\oiint d\Omega_{\mathbf{n}}\,(-4\boldsymbol{\beta}\cdot\mathbf{n})(\mathbf{n}\times\mathbf{a})\;=\;-\frac{16\pi}{3}(\boldsymbol{\beta}\times\mathbf{a}), \tag{S.129}$$

$$\begin{aligned}
\oiint d\Omega_{\mathbf{n}}\,(\boldsymbol{\beta}\times\mathbf{a}) &= 4\pi(\boldsymbol{\beta}\times\mathbf{a}), && \text{(S.130)}\\
\oiint d\Omega_{\mathbf{n}}\,(-\mathbf{n})(\mathbf{n}\cdot(\boldsymbol{\beta}\times\mathbf{a})) &= -\frac{4\pi}{3}(\boldsymbol{\beta}\times\mathbf{a}), && \text{(S.131)}\\
\text{altogether} &= -\frac{8\pi}{3}(\boldsymbol{\beta}\times\mathbf{a}), && \text{(S.132)}
\end{aligned}$$

and therefore

$$\frac{d\mathbf{L}_{\text{em}}}{dt_{\text{particle}}} = +\frac{Q^2\mu_0}{6\pi}\,(\boldsymbol{\beta}\times\mathbf{a}) = \frac{Q^2\mu_0}{6\pi c}\,(\mathbf{v}\times\mathbf{a}). \tag{10}$$

Quod erat demonstrangum.