Problem **1**:

The electric dipole moment of a hydrogen atom is simply $\hat{\mathbf{p}} = -e\hat{\mathbf{r}}$ in the coordinate system with the nucleus at the origin, hence

$$\langle 2|\,\hat{\mathbf{p}}\,|1\rangle \;=\; -e\iiint \Psi_2^*(\mathbf{r})\,\mathbf{r}\,\Psi_1(\mathbf{r})\,d^3\text{Vol}. \tag{S.1}$$

For the two states in questions, both wave functions (1) and (2) are invariant under rotations around the z axis, hence the matrix element (S.1) of the electric dipole moment must be parallel to that axis, thus

$$\langle 2|\,\hat{p}_x\,|1\rangle \;=\; \langle 2|\,\hat{p}_y\,|1\rangle \;=\; 0 \tag{S.2}$$

while

$$\langle 2|\,\hat{p}_z\,|1\rangle \;=\; -e\iiint \Psi_2^*(\mathbf{r})(z = r\cos\theta)\Psi_1(\mathbf{r})\,d^3\text{Vol}. \tag{S.3}$$

Evaluating this integral in spherical coordinates, we have

$$\begin{aligned}\Psi_2^*(r,\theta,\phi)\times(z=r\cos\theta)\times\Psi_1(r,\theta,\phi) \;&=\; \frac{1}{\sqrt{32\pi a^5}}\,re^{-r/2a}\cos\theta\times r\cos\theta\times\frac{1}{\sqrt{\pi a^3}}\,e^{-r/a}\\ &=\; \frac{1}{2^{5/2}\pi a^4}\times r^2e^{-3r/2a}\times\cos^2\theta,\end{aligned} \tag{S.4}$$

hence

$$\begin{aligned}\langle 2|\,\hat{p}_z\,|1\rangle \;&=\; -e\times\frac{1}{2^{5/2}\pi a^4}\times\int_0^\infty dr\,r^2\times r^2e^{-3r/2a}\times\int_{-1}^{+1} d\cos\theta\,\cos^2\theta\times\int_0^{2\pi} d\phi\\ &=\; -e\times\frac{1}{2^{5/2}\pi a^4}\times 24\left(\frac{2a}{3}\right)^5\times\frac{2}{3}\times 2\pi\\ &=\; -\left(\frac{8}{9}\right)^{5/2} ea.\end{aligned} \tag{S.5}$$

Classically, an oscillating electric dipole moment with amplitude $\mathbf{p}_0$ radiates net EM power

$$P \;=\; \frac{Z_0}{12\pi c^2}\,\omega^4|\mathbf{p}_0|^2. \tag{S.6}$$

In quantum mechanics, this formula corresponds to

$$P \;\equiv\; \hbar\omega\Gamma \;=\; \frac{Z_0}{12\pi c^2}\,\omega^4\;\left|2\,\langle 2|\,\hat{\mathbf{p}}\,|1\rangle\right|^2, \tag{S.7}$$

hence the transition rate is

$$\Gamma \;=\; \frac{Z_0}{3\pi c^2\hbar}\,\omega^3\,\left|\langle 2|\,\hat{\mathbf{p}}\,|1\rangle\right|^2. \tag{S.8}$$

For the $2p \to 1s$ transition in question,

$$\left|\langle 2|\,\hat{\mathbf{p}}\,|1\rangle\right|^2 \;=\; \left(\frac{8}{9}\right)^5 e^2a^2, \tag{S.9}$$

cf. eq. (S.5), while

$$\hbar\omega \;=\; E_1 \;-\; E_2 \;=\; -\frac{m_e e^4}{2(4\pi\epsilon_0\hbar)^2}\left(\frac{1}{4}-1\right) \;=\; +\frac{3}{8}\times\alpha^2 m_e c^2 \;=\; +\frac{3}{8}\,\frac{\alpha\hbar c}{a}\,, \tag{S.10}$$

hence

$$\Gamma \;=\; \frac{Z_0}{3\pi c^2\hbar^4}\times\frac{3^3}{2^9}\,(\alpha\hbar c/a)^3\;\times\frac{2^{15}}{3^{10}}\,e^2a^2 \;=\; \frac{2^6}{3^8\pi}\;\times\;\frac{Z_0e^2c\alpha^3}{\hbar a}\,. \tag{S.11}$$

Moreover,

$$Z_0e^2 \;=\; \frac{e^2}{\epsilon_0 c} \;=\; 4\pi\alpha\hbar, \tag{S.12}$$

hence

$$\Gamma \;=\; \frac{2^8}{3^8}\,\alpha^4\,\frac{c}{a}\,. \tag{S.13}$$

Numerically,

$$\frac{c}{a} \;\approx\; 5.66\cdot 10^{18}\ \mathrm{s}^{-1}, \qquad \alpha\approx\frac{1}{137}\,, \tag{S.14}$$

hence

$$\Gamma \;\approx\; 6.3\cdot 10^8\ \mathrm{s}^{-1}. \tag{S.15}$$

In other words, the average lifetime of the excited 2p state is $\Gamma^{-1}\approx 1.6$ ns.

Problem **2**:

In the electric dipole approximation, the allowed radiative transitions are the transitions between states with a non-zero matrix element of the electric dipole moment between them, $\langle 2|\,\hat{\mathbf{p}}\,|1\rangle \neq 0$. By the rotational and parity symmetries of the atomic states, such non-zero matrix elements are allowed only for

$$|j_1 - j_2| \;\leq\; 1 \;\leq\; j_1 + j_2\,, \quad |m_1^j - m_2^j| \;\leq\; 1, \quad \text{and} \quad \text{parity}_2 \;=\; -\text{parity}_1\,. \tag{S.16}$$

Moreover, for transitions involving a single electron — especially in a hydrogen-like atom or ion — between states with definite m^ℓ and m^s, we also need

$$|\ell_1 - \ell_2| \;\leq\; 1, \quad |m_1^\ell - m_2^\ell| \;\leq\; 1, \quad m_2^s \;=\; m_1^s, \tag{S.17}$$

and

$$(-1)^{\ell_2} \;=\; \text{parity}_2 \;=\; -\text{parity}_1 \;=\; -(-1)^{\ell_1}, \tag{S.18}$$

hence

$$\ell_2 \;=\; \ell_1 \;\pm\; 1 \quad (\text{but not } \ell_2 = \ell_1). \tag{S.19}$$

In this problem, we are going to ignore the electron's spin state — it's going to stay the same through the whole cascade of transitions, — and focus on the remaining quantum numbers n, ℓ and $m = m^\ell$.

Let's start with the state $|1\rangle = |n_1, \ell_1, m_1\rangle$ with $m_1 = \ell_1 = n_1 - 1$. The next state $|2\rangle = |n_2, \ell_2, m_2\rangle$ in the photon-emission cascade must have a lower energy than the initial state, which requires

$$n_2 \;<\; n_1 \tag{S.20}$$

and hence

$$m_2 \;\leq\; \ell_2 \;\leq\; n_2 - 1 \;\leq\; n_1 - 2. \tag{S.21}$$

On the other hand, by the selection rules (S.17), this state must have

$$m_2 \;\geq\; m_1 - 1 \quad \text{and} \quad \ell_2 \;\geq\; \ell_1 - 1, \tag{S.22}$$

which for the initial state with $m_1 = \ell_1 = n_1 - 1$ means

$$m_2, \ell_2 \ \geq\ n_1\ -\ 2. \tag{S.23}$$

Finally, the only way to reconcile the inequalities (S.21) and (S.23) is to have

$$m_2\ =\ \ell_2\ =\ n_2 - 1\ =\ n_1\ -\ 2. \tag{S.24}$$

Thus, in the electric dipole approximation to the photon emission, if the initial state has $m_1 = \ell_1 = n_1 - 1$ then the final state (of the first transition) is a similar state with $m_2 = \ell_2 = n_2 - 1$ for $n_2 = n_1 - 1$. *Quod erat demonstrandum.*

Problem **3**, preamble:

Far away from the antenna,

$$\mathbf{A}(r,\theta,\phi)\ \approx\ \mu_0\,\frac{e^{ikr}}{r}\,\mathbf{f}(\theta,\phi) \tag{S.25}$$

and hence the power (per solid angle) radiated by the antenna in the direction $\mathbf{n}$ is

$$\frac{dP}{d\Omega}\ =\ \frac{\omega^2 Z_0}{2c^2}\,|\mathbf{n}\times\mathbf{f}(\mathbf{n})|^2 \tag{S.26}$$

for

$$\mathbf{f}(\mathbf{n})\ =\ \frac{1}{4\pi}\iiint\limits_{\text{antenna}} d^3\mathbf{y}\,\mathbf{J}(\mathbf{y})\,\exp(-ik\mathbf{n}\cdot\mathbf{y}). \tag{S.27}$$

Eq. (S.26) for the power is exact — as long as the power is measured far away from the antenna — but the integral in eq. (S.27) often takes various approximations to calculate, for example, the multipole expansion for short antennas. However, in this problem we are going to calculate the exact $\mathbf{f}(\theta)$ for the antenna in question in part (a), and then compare it to the multipole expansion in later parts.

Problem $\mathbf{3}$(a):

For a thin antenna we approximate

$$\mathbf{J}(x,y,z) \;=\; \delta(x)\delta(y)I(z)\,\hat{\mathbf{z}} \tag{S.28}$$

where $\hat{\mathbf{z}}=(0,0,1)$ is the unit vector in the z direction, hence

$$\mathbf{f}(\mathbf{n}) \;=\; \frac{\hat{\mathbf{z}}}{4\pi}\int\!dz\,I(z)\,\exp(-ik(\mathbf{n}\cdot\hat{\mathbf{z}})z), \tag{S.29}$$

or in terms of the angle θ between the direction $\mathbf{n}$ towards the observer and the z axis,

$$\mathbf{f}(\theta) \;=\; \frac{\hat{\mathbf{z}}}{4\pi}\int\!dz\,I(z)\,\exp(-ik(\cos\theta)z). \tag{S.30}$$

For the antenna current as in eq. (12), the integral here evaluates to

$$\begin{aligned}
\mathbf{f}(\theta) \;&=\; \frac{\hat{\mathbf{z}}}{4\pi}\int\limits_{-L/2}^{+L/2} dz\,I_0\,\sin(kz)\,\exp(-i(k\cos\theta)z) \qquad \langle\!\langle\,\text{for } k=(2\pi/\lambda) \text{ and } L=\lambda\,\rangle\!\rangle\\
&=\; \frac{I_0\hat{\mathbf{z}}}{4\pi k}\int\limits_{-\pi}^{+\pi} dx\,\sin(x)\,\exp(-ix\cos\theta) \qquad \langle\!\langle\,\text{where } x=kz\,\rangle\!\rangle\\
&=\; \frac{I_0\hat{\mathbf{z}}}{8\pi ik}\int\limits_{-\pi}^{+\pi} dx\,\Big(\exp(i(1-\cos\theta)x)\;-\;\exp(i(-1-\cos\theta)x)\Big)\\
&=\; \frac{I_0\hat{\mathbf{z}}}{8\pi ik}\left[\frac{\exp(i(1-\cos\theta)x)}{i(1-\cos\theta)}\;-\;\frac{\exp(i(-1-\cos\theta)x)}{i(-1-\cos\theta)}\right]\Bigg|_{x=-\pi}^{x=+\pi}
\end{aligned} \tag{S.31}$$

where

$$\begin{aligned}
\exp(i(\pm1-\cos\theta)x)\Big|_{x=-\pi}^{x=+\pi} \;&=\; e^{\pm i\pi}e^{-i\pi\cos\theta}\;-\;e^{\mp i\pi}e^{+i\pi\cos\theta}\\
&=\; -e^{-i\pi\cos\theta}\;+\;e^{+i\pi\cos\theta}\;=\;2i\sin(\pi\cos\theta),
\end{aligned} \tag{S.32}$$

hence

$$\begin{aligned}
\mathbf{f}(\theta) \;&=\; \frac{I_0\hat{\mathbf{z}}}{8\pi ik}\left[\frac{2i\sin(\pi\cos\theta)}{i(1-\cos\theta)}\;-\;\frac{2i\sin(\pi\cos\theta)}{i(-1-\cos\theta)}\right]\\
&=\; \frac{I_0\hat{\mathbf{z}}}{8\pi ik}\,\sin(\pi\cos\theta)\left[\frac{2}{1-\cos\theta}\;-\;\frac{2}{-1-\cos\theta}\;=\;\frac{4}{1-\cos^2\theta}\right]\\
&=\; \frac{I_0\hat{\mathbf{z}}}{2\pi ik}\,\frac{\sin(\pi\cos\theta)}{\sin^2\theta}\,.
\end{aligned} \tag{S.33}$$

Note that as a vector, the $\mathbf{f}(\theta)$ always points in the $\hat{\mathbf{z}}$ direction, hence

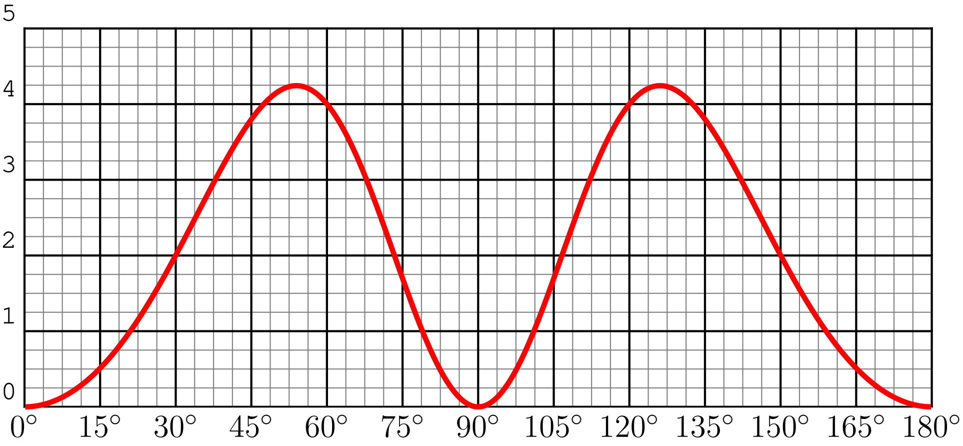
$$|\mathbf{n}\times\mathbf{f}|^2 \;=\; |\mathbf{f}|^2\times\sin^2\theta \;=\; \frac{|I_0|^2}{4\pi^2k^2}\times\frac{\sin^2(\pi\cos\theta)}{\sin^4\theta}\times\sin^2\theta \;=\; \frac{|I_0|^2}{4\pi^2k^2}\times\frac{\sin^2(\pi\cos\theta)}{\sin^2\theta}\,. \tag{S.34}$$

Consequently, the EM power (per solid angle) emitted in the direction $\mathbf{n}$ at angle θ from the z axis is

$$\begin{aligned}\frac{dP}{d\Omega} &= \frac{Z_0\omega^2}{2c^2}\,|\mathbf{n}\times\mathbf{f}|^2 \;=\; \frac{Z_0\omega^2}{2c^2}\,\frac{|I_0|^2}{4\pi^2k^2}\times\frac{\sin^2(\pi\cos\theta)}{\sin^2\theta}\\ &= \frac{Z_0|I_0|^2}{8\pi^2}\times\frac{\sin^2(\pi\cos\theta)}{\sin^2\theta}\,.\end{aligned} \tag{S.35}$$

Problem **3**(b):

Let's plot the angular distribution of the directional power (S.35).



This plot shows minima at the directions both parallel to the antenna's axis and also ⊥ to the axis. Indeed, for $\theta=\pi$,

$$\sin^2(\pi\cos\theta) \;=\; \sin^2(0) \;=\; 0, \tag{S.36}$$

while for $\theta\to 0$

$$\pi\cos\theta \;\approx\; \pi \,-\, \tfrac{\pi}{2}\,\theta^2 \quad\Longrightarrow\quad \sin(\pi\cos\theta) \;\approx\; \sin\bigl(\pi-\tfrac{\pi}{2}\theta^2\bigr) \;=\; \sin\bigl(\tfrac{\pi}{2}\theta^2\bigr) \;\approx\; \tfrac{\pi}{2}\,\theta^2 \tag{S.37}$$

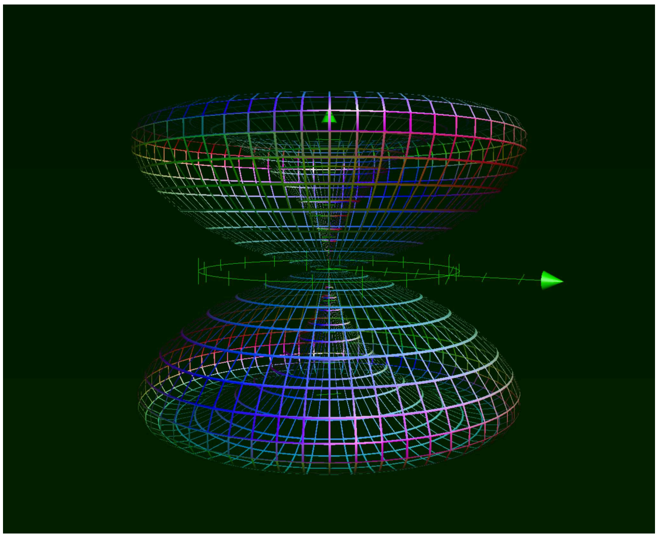
and hence

$$\frac{\sin^2(\pi\cos\theta)}{\sin^2\theta} \;\approx\; \frac{\pi^2}{4}\times\theta^2 \;\to\; 0, \tag{S.38}$$

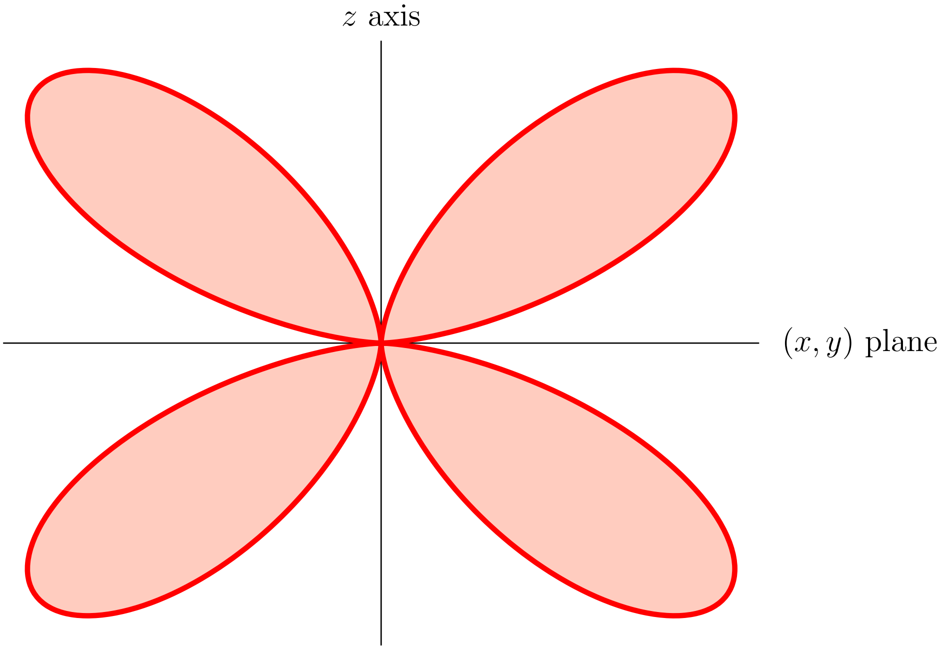
and likewise for $\theta\to\pi$.

Between these minima, the radiation is peaked at some intermediate angles; taking the numeric derivative of eq. (S.35), we find the power maxima at $\theta \approx 53.9^\circ$ and $\theta \approx 126.1^\circ$, both being 36.1° away from the (x, y) plane normal to the antenna.

For completeness sake, here is the radiation power diagram for this antenna: First, the 3D plot



and second, the cross-section through the antenna's axis:



(S.39)

Note that each conical lobe of the 3D diagram shows as a pair of distinct 'leaves' on this cross-section — on on the left side of the axis, and one on the right side, — thus 4 apparent leaves but only two lobes in 3D.

Problem **3**(c):

The net power emitted by the antenna in question obtains as an integral

$$\begin{aligned} P_{\rm net} &= \iint d^2\Omega \, \frac{dP}{d\Omega} = \frac{Z_0|I_0|^2}{8\pi^2} \times 2\pi \int_0^{\pi} d\theta \, \sin\theta \, \frac{\sin^2(\pi\cos\theta)}{\sin^2\theta} \\ &= \frac{Z_0|I_0|^2}{4\pi} \times \int_{-1}^{+1} dc \, \frac{\sin^2(\pi c)}{1-c^2} \end{aligned} \tag{S.40}$$

where on the last line I have changed the integration variable from θ to $c = \cos\theta$. Unfortunately, this integral does not evaluate in terms of elementary functions but only in terms of the cosine integral function

$$\mathrm{Cin}(x) \stackrel{\rm def}{=} \int_0^x \frac{1-\cos(t)}{t} \, dt. \tag{S.41}$$

Indeed,

$$\frac{\sin^2(2\pi c)}{1-c^2} = \frac{1}{4}\bigl(1-\cos(2\pi c)\bigr)\left(\frac{1}{1-c} + \frac{1}{1+c}\right) \tag{S.42}$$

hence

$$\begin{aligned} \int_{-1}^{+1} dc \, \frac{\sin^2(\pi c)}{1-c^2} &= \frac{1}{4} \int_{-1}^{+1} dc \left(\frac{1-\cos(2\pi c)}{1-c} + \frac{1-\cos(2\pi c)}{1+c}\right) \\ &\qquad \langle\!\langle \text{ using } c \to -c \text{ symmetry } \rangle\!\rangle \\ &= \frac{1}{2} \int_{-1}^{+1} dc \, \frac{1-\cos(2\pi c)}{1+c} \\ &\qquad \langle\!\langle \text{ changing variable from } c \text{ to } t = 2\pi(1+c) \rangle\!\rangle \\ &= \frac{1}{2} \int_0^{4\pi} dt \, \frac{1-\cos(t)}{t} \\ &= \tfrac{1}{2}\,\mathrm{Cin}(4\pi), \end{aligned} \tag{S.43}$$

and therefore

$$P_{\text{net}} = \frac{\text{Cin}(4\pi)}{4\pi} \times \frac{Z_0|I_0|^2}{2}\,. \tag{S.44}$$

In terms of the antenna's radiative resistance, this means

$$R_{\text{rad}} = \text{Re}(Z_{\text{rad}}) = \frac{\text{Cin}(4\pi)}{4\pi} \times Z_0\,. \tag{S.45}$$

Numerically,

$$\frac{\text{Cin}(4\pi)}{4\pi} \approx 0.248, \tag{S.46}$$

thus

$$R_{\text{rad}} \approx 0.248\,Z_0 \approx 93.4\ \Omega. \tag{S.47}$$

Problem **4**:

An accelerating charged particle radiates EM energy, and the back reaction of this radiation on the particle's motion acts as an affective reaction force. For a non-relativistic particle, this reaction force is given by the *Abraham–Lorentz formula*

$$\mathbf{F}_{\text{rad}} = \frac{\mu_0 q^2}{6\pi c}\,\dot{\mathbf{a}}, \tag{S.48}$$

cf. textbook eq. (11.80). Between this reaction force and the potential force $F = -m\omega_0^2 \times x$, the electron's equation of motion is

$$ma = F_{\text{net}} = -m\omega_0^2 \times x + \frac{\mu_0 e^2}{6\pi c} \times \dot{a}, \tag{S.49}$$

or in terms of electron's position x and its derivatives,

$$\tau\,\dddot{x} - \ddot{x} - \omega_0^2 x = 0, \tag{S.50}$$

where

$$\tau = \frac{\mu_0 e^2}{6\pi c m_e} \approx 6.27 \cdot 10^{-24}\ \text{s}. \tag{S.51}$$

Eq. (S.50) is a linear third-order differential equation with constant coefficients, so its general solution is a linear combination of three exponentials,

$$x(t) \;=\; \sum_{k=1,2,3} A_k \exp(\nu_k t), \tag{S.52}$$

where ν_1, ν_2, ν_3 are roots of the cubic equation

$$\tau \times \nu^3 \;-\; \nu^2 \;-\; \omega_0^2 \;=\; 0. \tag{S.53}$$

Assuming $\omega_0\tau \ll 1$, we may approximate these 3 roots as

$$\begin{aligned}
\nu_1 &= +i\omega_0 \;-\; \tfrac{1}{2}\omega_0^2\tau \;-\; \tfrac{5}{8}i\omega_0^3\tau^2 \;+\; \omega_0^4\tau^3 \;+\; \tfrac{231}{128}i\omega_0^5\tau^4 \;+\; \cdots, \\
\nu_2 &= -i\omega_0 \;-\; \tfrac{1}{2}\omega_0^2\tau \;+\; \tfrac{5}{8}i\omega_0^3\tau^2 \;+\; \omega_0^4\tau^3 \;-\; \tfrac{231}{128}i\omega_0^5\tau^4 \;+\; \cdots, \\
\nu_3 &= \frac{1}{\tau} \;+\; \omega_0^2\tau \;-\; 2\omega_0^4\tau^3 \;+\; \cdots.
\end{aligned} \tag{S.54}$$

Note: the ν_3 root is real, positive, and large, so the corresponding solution $x(t) \propto \exp(\nu_3 t)$ is the un-physical runaway. Such unphysical runaway solutions are a common problem of equations of motion involving the Abraham–Lorentz reaction force, and the only thing we can do with such solutions is to simply reject them. Thus, the only physical solutions are linear combinations of the other two exponentials $\exp(\nu_1 t)$ and $\exp(\nu_2 t)$.

Unlike the ν_3 root, the other two roots $\nu_{1,2}$ are complex — and complex-conjugate to each other — with a small but negative real part:

$$\begin{aligned}
\nu_{1,2} &= \pm i\omega \;-\; \tfrac{1}{2}\gamma \quad \text{for} \\
\omega &= \omega_0 \;-\; \tfrac{5}{8}\omega_0^3\tau^2 \;+\; \tfrac{231}{128}\omega_0^5\tau^4 \;+\; \cdots, \\
\gamma &= \omega_0^2\tau \;-\; 2\omega_0^4\tau^3 \;+\; \cdots.
\end{aligned} \tag{S.55}$$

Consequently, the general physical solution of the equation of motion (S.50) is

$$\begin{aligned}
x(t) &= \mathrm{Re}\Big(\hat{A}e^{-i\omega t}e^{-\gamma t/2}\Big) \\
&= A\cos(\omega t)e^{-\gamma t/2} \;+\; B\sin(\omega t)e^{-\gamma t/2}
\end{aligned} \tag{S.56}$$

for some real amplitudes A and B (or equivalently, complex amplitude $\hat{A} = A + iB$). Clearly, these solutions describe *dumped harmonic oscillations* of angular frequency $\omega \approx \omega_0$ and decay rate $\gamma \approx \omega_0^2\gamma$.

Finally, the quality factor of a (dumped) harmonic oscillator is

$$Q = \frac{\omega_0}{\gamma}, \tag{S.57}$$

cf. homework set#2, problem 4. For the electron at hand,

$$Q \approx \frac{\omega_0}{\gamma \approx \omega_0^2 \tau} = \frac{1}{\omega_0 \tau}. \tag{S.58}$$

For example, for the ω_0 in the visible-light frequency range $(2.5 \text{ to } 5) \cdot 10^{15}\ \mathrm{s}^{-1}$, Q is in the 32 million to 64 million range.

Problem **4**(a):

For the electron in a harmonic potential $\mathbf{F}_{\mathrm{ext}} = -m\omega_0^2 \times x$, so eq. (7) becomes

$$ma = \left(1 + \tau\frac{d}{dt}\right)(-m\omega_0^2 \times x) = -m\omega_0^2 \times \Big(x + \tau v\Big) \tag{S.59}$$

and hence

$$a + \tau\omega_0^2 \times v + \omega_0^2 \times x = 0. \tag{S.60}$$

Or in manifest terms of $x(t)$ and its time derivatives,

$$\ddot{x} + \tau\omega_0^2 \times \dot{x} + \omega_0^2 \times x = 0. \tag{S.61}$$

This is a second-order differential equation with constant coefficients, so it has two independent solutions $x(t) = A \times \exp(\nu t)$ where ν is a root of the quadratic equation

$$\nu^2 + \tau\omega_0^2 \times \nu + \omega_0^2 = 0, \tag{S.62}$$

hence

$$\nu = \pm i\omega - \frac{\gamma}{2} \tag{S.63}$$

where

$$\gamma \;=\; \tau\omega_0^2 \tag{S.64}$$

while

$$\omega \;=\; \sqrt{\omega_0^2 - \tfrac{1}{4}\gamma^2} \;\approx\; \omega_0\,. \tag{S.65}$$

Consequently, the general solution to the Newton equation is the dumped oscillation

$$\begin{aligned} x(t) \;&=\; \mathrm{Re}\Big(\hat{A}e^{-i\omega t}e^{-\gamma t/2}\Big) \\ &=\; A\cos(\omega t)e^{-\gamma t/2} \;+\; B\sin(\omega t)e^{-\gamma t/2} \end{aligned} \tag{S.66}$$

with frequency $\omega \approx \omega_0$ and decay rate $\gamma = \tau\omega_0^2$. The quality factor of this damped oscillator is

$$Q \;=\; \frac{\omega_0}{\gamma} \;=\; \frac{1}{\tau\omega_0}\,, \tag{S.67}$$

same as in problem 4. However, unlike what we had in problem 4, this time we do not have any unphysical runaway solutions $x \propto \exp(+\text{large} \times t)$.

Problem **4**(b):

The force (8) can be described in terms of the step function Θ as

$$F_{\text{ext}}(t) \;=\; F_0\Theta(t)\Theta(T-t). \tag{S.68}$$

The derivative of a step function $\Theta(t)$ is the delta-function $\delta(t)$, hence

$$\frac{dF_{\text{ext}}}{dt} \;=\; F_0\delta(t)\Theta(T-t) \;-\; F_0\Theta(t)\delta(T-t) \;=\; F_0\delta(t) \;-\; F_0\delta(t-T). \tag{S.69}$$

Consequently, eq. (7) for the electron subject to this force is

$$m\times a(t) \;=\; F_0\times\Theta(t)\Theta(T-t) \;+\; \tau F_0\times\big(\delta(t)-\delta(t-T)\big). \tag{S.70}$$

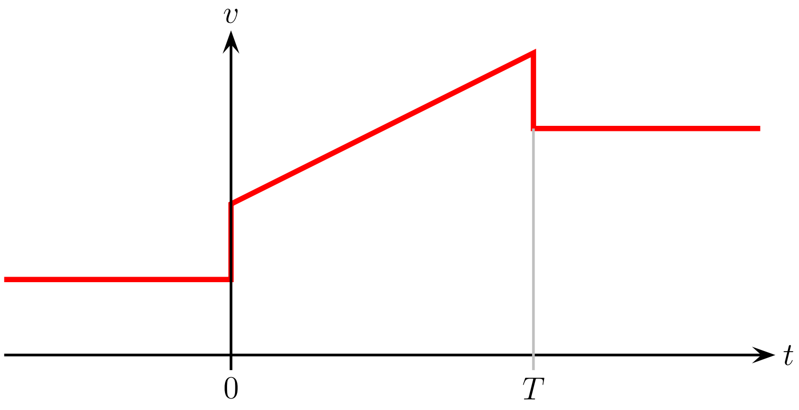
The RHS here does not depend on the electron's position of velocity, so it is easy to integrate. Integrating once for the velocity $v(t)$, we get:

- At $t < 0$ the electron moves at constant velocity $v(t) \equiv v_0$.
- At $t = 0$, the electron suddenly accelerates by $\Delta v = \tau F_0/m$, thus $v(t = +0) = v_0 + (\tau F_0/m)$.
- At $t > 0$ but $t < T$, the electron has constant acceleration $a = F_0/m$, hence

$$v(t) \;=\; v_0 \;+\; \frac{\tau F_0}{m} \;+\; \frac{F_0}{m} \times t. \tag{S.71}$$

- At $t = T$ the electron suddenly decelerates by $\Delta v = -\tau F_0/m$, thus $v(t = T + 0) = v_0 + (F_0/m)T$.
- At $t > T$ the electron's acceleration ceases and it continues at constant velocity $v = v_0 + (F_0/m)T$.

Here is the plot:

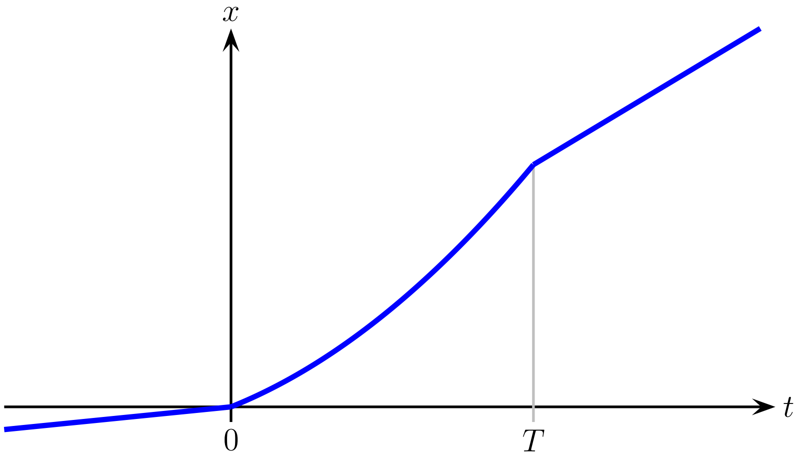


(S.72)

Finally, integrating the second time to get $x(t)$, we get:

$$\begin{aligned}
\text{for } t < 0: \quad & x(t) \;=\; x_0 \;+\; v_0 \times t, && \text{(S.73)}\\
\text{for } 9 < t < T: \quad & x(t) \;=\; x_0 \;+\; \left(v_0 + \frac{\tau F_0}{m}\right) \times t \;+\; \frac{F_0}{2m} \times t^2, && \text{(S.74)}\\
\text{for } t > T: \quad & x(t) \left(x_0 \;+\; \frac{\tau F_0}{m} T + \frac{F_0}{2m} T^2)\right) \;+\; \left(v_0 \;+\; \frac{F_0}{m} T\right) \times t. && \text{(S.75)}
\end{aligned}$$

Graphically,



(S.76)