Problem **1**:

A half-dipole short vertical antenna radiates directional power

$$\frac{dP}{d\Omega} = \frac{Z_0\omega^2 h^2 I_0^2}{32\pi^2 c^2} \times \sin^2\theta \tag{S.1}$$

cf. my notes (eqs. (6) through (63)), but only above the ground, for $\theta \leq 90°$. Hence, the net radiated power is

$$P = \frac{Z_0\omega^2 h^2 I_0^2}{32\pi^2 c^2} \times \frac{4\pi}{4} = \frac{Z_0\omega^2 h^2 I_0^2}{24\pi c^2}, \tag{S.2}$$

while the antenna's directivity is

$$D(\theta) = \frac{dP}{d\Omega} \Bigg/ \frac{P}{4\pi} = 3\sin^2\theta. \tag{S.3}$$

Consequently, given the net power P radiated by the transmitting antenna, the wave power flux at the second antenna's location — which is at distance R from the first antenna in the horizontal direction $\theta = 90°$, — is

$$I = \frac{P}{4\pi R^2} \times D(90°) = \frac{P}{4\pi R^2} \times 3. \tag{S.4}$$

Next consider the receiving antenna. If it has a properly matched tuner, then the effective aperture of the antenna is

$$A = \frac{\lambda^2}{4\pi} \times D(\theta) \times \cos^2\psi, \tag{S.5}$$

cf. my notes on the receiving antennas. In this formula, the directivity $D(\theta)$ is the same as for the transmitted antenna, thus for two vertical antennas at the same elevation $\theta = 90°$ and $D(90°) = 3$. Finally, ψ is the angle between the vertical axis of the antenna and the plane of the wave's polarization. Since the transmitting antenna is also vertical, the wave it broadcast is vertically polarized, thus $\psi = 0$ and $\cos^2\psi = 1$. Altogether,

$$A = \frac{\lambda^2}{4\pi} \times 3. \tag{S.6}$$

Consequently, the power received by the antenna and going to the tuner is

$$P_{\rm rec} = I \times A = \frac{3P}{4\pi^2 R} \times \frac{3\lambda^2}{4\pi} = P \times \frac{9\lambda^2}{16\pi^2 R^2}\,. \tag{S.7}$$

Or as a fraction of the net broadcast power

$$\frac{P_{\rm rec}}{P} = \frac{9\lambda^2}{16\pi^2 R^2}. \tag{S.8}$$

For the problem at hand $\lambda = 60$ m while $R = 20$ km, hence

$$\frac{9\lambda^2}{16\pi^2 R^2} = \left(\frac{3 \times 60\ \text{m}}{4\pi \times 20,000\ \text{m}}\right)^2 = 5.13 \cdot 10^{-7} \approx 0.5 \cdot 10^{-6}. \tag{S.9}$$

Thus, if the first antenna broadcasts at 1 kW power, then the second antenna (or rather its tuner) receives only 0.5 mW of that power.

Problem **2**(a):

The velocities $\mathbf{v}_1$ and $\mathbf{v}_2$ of the same body relative to the two frames of reference $\mathcal{S}_1$ and $\mathcal{S}_2$ are related by the Galilean formula

$$\mathbf{v}_2 = \mathbf{v}_1 + \mathbf{u}_{12}\,, \tag{1}$$

so given $\mathbf{v}_1$ and $\mathbf{v}_2$, the velocity on one frame relative to the other frame obtains as

$$\mathbf{u}_{12} = \mathbf{v}_2 - {}_{\mathbf{1}}\,. \tag{S.10}$$

If both frames $\mathcal{S}_1$ and $\mathcal{S}_2$ are inertial, then for any body free from all forces the velocities $\mathbf{v}_1$ and $\mathbf{v}_2$ are constant. Consequently, $\mathbf{v}_2 - \mathbf{v}_1 = \text{const}$, so the relative velocity $\mathbf{u}_{12}$ of the two frames must be constant. *Quod erat demonstrandum.*

Conversely, if the relative velocity of the two frames is constant, then any body with a constant velocity $\mathbf{v}_1$ relative to one frame also has a constant velocity $\mathbf{v}_2$ relative to the other frame. So if the frame $\mathcal{S}_1$ happens to be inertial, then for any free body $\mathbf{v}_1 = \text{const}$, hence $\mathbf{v}_2 = \text{const}$, and that makes the $\mathcal{S}_2$ frame also inertial. *Quod erat demonstrandum.*

Problem **2**(b):

The net momentum of the two initial particles relative to the frame $\mathcal{S}_1$ is

$$\mathbf{p}_1 \;=\; m_A\mathbf{v}_{A1} \;+\; m_B\mathbf{v}_{B1}\,, \tag{S.11}$$

while relative to the frame $\mathcal{S}_2$ the net momentum is

$$\begin{aligned} \mathbf{p}_2 \;&=\; m_A\mathbf{v}_{A2} \;+\; m_B\mathbf{v}_{B2} \\ &\qquad \langle\!\langle \text{ using eq. (1) for each particle} \rangle\!\rangle \\ &=\; m_A(\mathbf{v}_{A1}+\mathbf{u}_{12}) \;+\; m_B(\mathbf{v}_{B1}+\mathbf{u}_{12}) \\ &=\; \Big(m_A\mathbf{v}_{A2} \;+\; m_B\mathbf{v}_{B2}\Big) \;+\; (m_A+m_b)\mathbf{u}_{12} \\ &=\; \mathbf{p}_1 \;+\; m_{\text{net}}\mathbf{u}_{12}\,. \end{aligned} \tag{S.12}$$

Likewise, for the final-state particles, the net momentum relative to the first frame $\mathcal{S}_1$ is

$$\mathbf{p}'_1 \;=\; m'_A\mathbf{v}'_{A1} \;+\; m'_B\mathbf{v}'_{B1} \tag{S.13}$$

and relative to the second frame $\mathcal{S}_2$ it's

$$\mathbf{p}'_2 \;=\; \mathbf{p}'_1 \;+\; m'_{\text{net}}\mathbf{u}_{12}\,. \tag{S.14}$$

But the net mass is conserved, *cf.* eq. (2), hence

$$m'_{\text{net}}\mathbf{u}_{12} \;=\; m_{\text{net}}\mathbf{u}_{12} \tag{S.15}$$

and therefore

$$\textbf{if}\quad \mathbf{p}'_1 \;=\; \mathbf{p}_1 \quad\textbf{then}\quad \mathbf{p}'_2 \;=\; \mathbf{p}_2 \quad\textbf{and vice verse}, \tag{S.16}$$

thus if the net momentum is conserved in one inertial frame then it's also conserved in any other inertial frame. *Quod erat demonstrandum.*

Problem **2**(c):

The net kinetic energy of the two particles in frame $\mathcal{S}_1$ is

$$U_1 \;=\; \tfrac{1}{2}m_A v_{A1}^2 \;+\; \tfrac{1}{2}m_B v_{B1}^2\,, \tag{S.17}$$

while in frame $\mathcal{S}_2$ it's

$$\begin{aligned}
U_2 \;&=\; \tfrac{1}{2}m_A v_{A2}^2 \;+\; \tfrac{1}{2}m_B v_{B2}^2\\
&\qquad\langle\!\langle\,\text{using eq. (1) for each particle}\,\rangle\!\rangle\\
&=\; \tfrac{1}{2}m_A(\mathbf{v}_{A1}+\mathbf{u}_{12})^2 \;+\; \tfrac{1}{2}m_B(\mathbf{v}_{B1}+\mathbf{u}_{12})^2\\
&=\; \tfrac{1}{2}m_A v_{A1}^2 \;+\; m_A\mathbf{v}_{A1}\cdot\mathbf{u}_{12} \;+\; \tfrac{1}{2}m_A u_{12}^2 \;+\tfrac{1}{2}m_B v_{B1}^2 \;+\; m_B\mathbf{v}_{B1}\cdot\mathbf{u}_{12} \;+\; \tfrac{1}{2}m_B u_{12}^2\\
&=\; \left(\tfrac{1}{2}m_A v_{A1}^2 \;+\; \tfrac{1}{2}m_B v_{B1}^2\right) \;+\; \Big(m_A\mathbf{v}_{A1} \;+\; m_B\mathbf{v}_{B1}\Big)\cdot\mathbf{u}_{12} \;+\; \tfrac{1}{2}\big(m_A+m_B\big)u_{12}^2\\
&=\; U_1 \;+\; \mathbf{p}_1\cdot\mathbf{u}_{12} \;+\; \tfrac{1}{2}m_{\text{net}}u_{12}^2\,.
\end{aligned} \tag{S.18}$$

Likewise, for the final-state particles

$$U_1' \;=\; \tfrac{1}{2}m_A' v_{A1}'^2 \;+\; \tfrac{1}{2}m_B v_{B1}'^2 \tag{S.19}$$

while

$$U_2' \;=\; U_1' \;+\; \mathbf{p}_1'\cdot\mathbf{u}_{12} \;+\; \tfrac{1}{2}m_{\text{net}}' u_{12}^2\,. \tag{S.20}$$

According to eqs. (2) and (3), the net mass and the net momentum are conserved in the collision, $m_{\text{net}}' = m_{\text{net}}$ and $\mathbf{p}_1' = \mathbf{p}_1$, hence in light of eqs. (S.18) and (S.20):

$$U_2' \;-\; U_1' \;=\; \mathbf{p}_1'\cdot\mathbf{u}_{12} \;+\; \tfrac{1}{2}m_{\text{net}}' u_{12}^2 \;=\; \mathbf{p}_1\cdot\mathbf{u}_{12} \;+\; \tfrac{1}{2}m_{\text{net}}u_{12}^2 \;=\; U_2 \;-\; U_1 \tag{S.21}$$

and therefore

$$U_2' \;-\; U_2 \;=\; U_1' \;-\; U_1\,. \tag{S.22}$$

Thus, IF the kinetic energy is conserved in the $\mathcal{S}_1$ frame, $U_1' = U_1$, THEN it must also be conserved in the $\mathcal{S}_2$ frame, $U_2' = U_2$. *Quod erat demonstrandum.*

Problem **3**(a):

By the Galilean rule, the ground velocity of the plane is simply the vector sum of its air-velocity and the wind-velocity; in particular, in a tailwind

$$v_{\text{gr}} \;=\; v_{\text{air}} \;+\; u_{\text{wind}} \;=\; 250\text{ m/s} \;+\; 50\text{ m/s} \;=\; 300\text{ m/s}. \tag{S.23}$$

The Einstein's rule (4) is more complicated, so the error in using the Galilean rule is

$$\begin{aligned}\Delta v \;&=\; v_{\text{gr}}[\text{Galileo}] \;-\; v_{\text{gr}}[\text{Einstein}]\\ &=\; (v_{\text{air}} + u_{\text{wind}}) \;-\; \frac{(v_{\text{air}} + u_{\text{wind}})}{1 \;+\; (v_{\text{air}}u_{\text{wind}})/c^2}\\ &=\; (v_{\text{air}} + u_{\text{wind}}) \times \frac{v_{\text{air}}u_{\text{wind}}}{c^2 \;+\; v_{\text{air}}u_{\text{wind}}}\,.\end{aligned} \tag{S.24}$$

For the plane at hand,

$$(v_{\text{air}} + u_{\text{wind}}) \;=\; 300\text{ m/s}, \tag{S.25}$$

$$v_{\text{air}} \times u_{\text{wind}} \;= 12,500\text{ m}^2/\text{s}^2\; 1.25 \cdot 10^4\text{ m}^2/\text{s}^2, \tag{S.26}$$

$$c^2 \;=\; 9 \cdot 10^{16}\text{ m}^2/\text{s}^2, \tag{S.27}$$

hence

$$\frac{v_{\text{air}} \times u_{\text{wind}}}{c^2 \;+\; v_{\text{air}} \times u_{\text{wind}}} \;\approx\; \frac{v_{\text{air}} \times u_{\text{wind}}}{c^2} \;\approx\; 1.39 \cdot 10^{-13}, \tag{S.28}$$

and therefore

$$\Delta v \;=\; (300\text{ m/s}) \times 1.39 \cdot 10^{-13} \;=\; 4.17 \cdot 10^{-11}\text{ m/s}, \tag{S.29}$$

or about 0.15 micrometers per hour.

Obviously, errors like this can be safely neglected. And that's why we keep using the Galilean formula in our everyday lives.

Problem **3**(b):

Suppose $|v_1| < c$ and $|u| < c$, then

$$c^2 \mp c(v_1+u) + v_1u = (c\mp v_1)(c\mp u) > 0 \tag{S.30}$$

for either sign $\mp$, and therefore

$$c\times|v_1+u| < c^2 + v_1u\,. \tag{S.31}$$

Consequently,

$$|v_2| = \left|\frac{v_1+u}{1+(v_1u)/c^2}\right| = \frac{c^2\times|v_1+u|}{c^2+v_1u} < c. \tag{S.32}$$

Quod erat demonstrandum.

Problem **3**(c):

Using the ground as frame#2, and the train — with the shooter standing in it — as frame#1, we have relative velocity u_{12} = train's velocity $= \frac{1}{2}c$, while the bullet's velocity relative to the gun — and the train — is $v_1 = \frac{2}{3}c$. Consequently, the bullet's velocity relative to the ground obtains from eq. (4):

$$v_2 = \frac{v_1+u_{12}}{1+(v_1/c)\times(u_{12}/c)} = \frac{\frac{1}{2}c + \frac{2}{3}c}{1 + \frac{1}{2}\times\frac{2}{3}} = \frac{\frac{7}{6}c}{\frac{4}{3}} = \frac{7}{8}c. \tag{S.33}$$

Problem **4**(a):

During the *retarded time* $\Delta t_{\rm ret}$, the star (or some other luminous object) moves through distance $v\delta t_{\rm ret}$, including

$$-\Delta\mathcal{R} = v\Delta t_{\rm ret}\times\cos\theta \tag{S.34}$$

towards the observer on Earth and

$$\Delta s = v\Delta t_{\rm ret}\times\sin\theta \tag{S.35}$$

across the observer's sky. As to the observer's time

$$t = t_{\rm ret} + \frac{\mathcal{R}}{c}, \tag{S.36}$$

it increases by

$$\Delta t = \Delta t_{\rm ret} + \frac{\Delta\mathcal{R}}{c} = \Delta t_{\rm ret} - \frac{v\Delta t_{\rm ret}\times\cos\theta}{c} = \Delta t_{\rm ret}\times\left(1-\frac{v}{c}\times\cos\theta\right). \tag{S.37}$$

Consequently, the *apparent velocity* of the star across the sky is

$$v_{\rm app} = \frac{\Delta s}{\Delta t} = \frac{v\sin\theta}{1-(v/c)\cos\theta}. \tag{S.38}$$

Problem **4**(b):

To find the maximum of the apparent velocity (S.38) as a function of θ at a fixed v, we take the derivative

$$\begin{aligned}
\frac{\partial v_{\rm app}}{\partial\theta} &= \frac{v\cos\theta}{[1-(v/c)\cos\theta]} - \frac{v\sin\theta}{[1-(v/c)\cos\theta]^2}\times\left(\frac{\partial-(v/c)\cos\theta}{\partial\theta} = +(v/s)\sin\theta\right)\\
&= \frac{v}{[1-(v/c)\cos\theta]^2}\times\left(\cos\theta[1-(v/c)\cos\theta] - (v/c)\sin^2\theta\right)\\
&= \frac{v}{[1-(v/c)\cos\theta]^2}\times\left(\cos\theta - (v/c)\right).
\end{aligned} \tag{S.39}$$

This derivative vanishes for $\cos\theta = (v/c)$, and it's easy to see that this stationary point is indeed a maximum. Thus, the greatest apparent velocity of a star obtains for

$$\theta[\max] = \arccos\frac{v}{c}. \tag{S.40}$$

At this angle,

$$\sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-(v/c)^2}, \tag{S.41}$$

$$-1 - (v/c)\cos\theta = [1-(v/c)^2], \tag{S.42}$$

and therefore

$$v_{\rm app} = \frac{v\sin\theta}{[1-(v/c)\cos\theta]} = \frac{v}{\sqrt{1-(v/c)^2}}\,. \qquad \text{(S.43)}$$

Or in terms of the relativistic parameters $\beta = (v/c)$ and $\gamma = 1/\sqrt{1-\beta^2}$,

$$v_{\rm app} = \gamma\beta c. \qquad \text{(S.44)}$$

<u>Problem **4**(c)</u>:

Unlike the actual velocity of the star that's always slower that c, the apparent velocity (S.44) can grow as large as we wish as v gets closer and closer to c. In particular, the apparent velocity can reach c for $\beta\gamma = 1$, which calls for

$$\beta^2\gamma^2 = \frac{\beta^2}{1-\beta^2} = 1 \implies \beta^2 = \frac{1}{2} \implies v = \frac{c}{\sqrt{2}}\,, \qquad \text{(S.45)}$$

and for larger v, the apparent velocity becomes superluminal. (Assuming the star moves in the right direction.) And for v asymptoting to the speed of light, — say, $c = c\times(1-\epsilon)$ for a very small ϵ, — we have $\beta = 1-\epsilon$, $\beta\gamma \approx 1/\sqrt{2\epsilon}$ and hence

$$v_{\rm app} \approx \frac{c}{\sqrt{2\epsilon}} \xrightarrow[\epsilon\to+0]{} \infty. \qquad \text{(S.46)}$$

<u>Problem **5**(a)</u>:

The pion's average lifetime is $\Delta t_{\rm avg} = \tau = 26$ ns *in the frame where the pion is at rest.* But when the pion flies with a speed comparable to the speed of light, it's own proper time slows down relative to the lab frame by the factor of $1/\gamma = \sqrt{1-(v/c)^2}$. Consequently, *in the lab frame, the average lifetime of the* π^+ *meson is not* $\tau = 26$ *ns but*

$$\Delta t_{\rm avg} = \tau\times\gamma > \tau. \qquad \text{(S.47)}$$

Problem $\mathbf{5}$(b):

Assuming all pions in the beam fly with the same velocity v, the average distance they fly before decaying is

$$\Delta L_{\rm avg} \;=\; v\times\Delta t_{\rm avg} \;=\; v\times\gamma\times\tau. \tag{S.48}$$

Consequently, for the given data $\tau=26$ ns and $\Delta L_{\rm avg}=13$ m, we find

$$v\times\gamma \;=\; \frac{\Delta L_{\rm avg}}{\tau} \;=\; 5\cdot 10^8\ {\rm m/s} \tag{S.49}$$

instead of eq. (6).

To solve eq. (S.49) for the velocity, we use

$$(v\times\gamma)^2 \;=\; c^2\times\beta^2\gamma^2 \;=\; c^2\times\frac{\beta^2}{1-\beta^2}\,, \tag{S.50}$$

hence

$$\frac{\beta^2}{1-\beta^2} \;=\; \left(\frac{5\cdot 10^8\ {\rm m/s}}{c}\right)^2 \;=\; 2.78, \tag{S.51}$$

hence

$$\beta^2 \;=\; 0.736 \quad\Longrightarrow\quad \beta \;=\; 0.858 \tag{S.52}$$

and therefore $v=2.57\cdot 10^8$ m/s.

PS: The originally assigned homework has a typo: It gave $\Delta L_{\rm avg}=6.5$ m instead of 13 m. Consequently, eq. (S.51) yields

$$\frac{\beta^2}{1-\beta^2} \;=\; \left(\frac{2.5\cdot 10^8\ {\rm m/s}}{c}\right)^2 \;=\; 0.695, \tag{S.53}$$

hence $\beta=0.64$ and $v=1.92\cdot 10^8$ m/s.

For the grading purposes, both $\beta\approx 0.86$ and $\beta\approx 0.64$ are accepted as correct.

Problem **6**(a):

In the frame of the Earth, the shipboard clock runs slower than the Earthbound clocks by the factor of

$$\frac{1}{\gamma} = \sqrt{1-\beta^2} = \sqrt{1-(\tfrac{4}{5})^2} = \tfrac{3}{5}. \tag{S.54}$$

The signal was sent at $t_s = 1$ hour by the slower shipboard clock. By the Earthbound clock, this time was

$$t_E = \gamma \times t_s = \tfrac{5}{3} \times 1 \text{ hr} = 100 \text{ min}. \tag{S.55}$$

Problem **6**(b):

By the time the ship sent its radio signal, the ship was at a distance from Earth — as measured in the Earth frame —

$$L_E = v \times t_E = \tfrac{4}{5}c \times 100 \text{ min} = 80 \text{ light-minutes}. \tag{S.56}$$

The radio signal takes 80 minutes to travel this distance, so by the time it arrives the Earthbound clock shows

$$t_E = 100 \text{ min} + 80 \text{ min} = 180 \text{ min} = 3 \text{ hr}. \tag{S.57}$$

Problem **6**(c):

In the ship's frame, it's the Earthbound clock that runs slower than the ship's clock by the factor of $1/\gamma = \frac{3}{5}$. In part (b) we saw that the Earth clock shows $t_E = 3$ hours when the signal arrives, by the ship's clock this time is

$$t_s = \gamma \times t_E = \tfrac{5}{3} \times 3 \text{ hr} = 5 \text{ hr}. \tag{S.58}$$

Alternative solution:

In the ship's frame, the distance to Earth is Lorentz-contracted by the factor $1/\gamma$, so at the time the ship sent the radio signal the Earth was

$$L_s \;=\; \frac{1}{\gamma}\times L_E \;=\; \frac{3}{5}\times 80\text{ light-minutes} \;=\; 48\text{ light-minutes}. \tag{S.59}$$

The radio signal travels at the speed of light, but it has to chase the Earth which recedes at the speed $V=\frac{4}{5}c$, so the signal travels through time

$$\Delta t_s \;=\; \frac{L_s}{c-v} \;=\; \frac{48\text{ light-minutes}}{\frac{1}{5}c} \;=\; 240\text{ min} \;=\; 4\text{ hr}. \tag{S.60}$$

Adding this signal travel time to $t_s=1$ hour when the signal was sent, we find that *by the ship-board clock, the signal arrives to Earth at* $t_s=5$ hours.

Problem **7**:

Let L_A and L_B are the lengths of the two ships when measured in the rest frame of each ship; Presuming the Galactic Catalogue is correct, we know that $L_A=2L_B$.

But when the ships fly through the solar system at relativistic speeds, their respective length in the Earth frame become Lorentz-shortened by the respective factors $1/\gamma=\sqrt{1-\beta^2}$, thus

$$L'_A \;=\; \frac{L_A}{\gamma_A}\quad\text{and}\quad L'_B \;=\; \frac{L_B}{\gamma_B}\,. \tag{S.61}$$

Note that the two ships have different speeds and hence $\gamma_A\neq\gamma_B$, and that's how the two ships end up with similar lengths $L'_A=L'_B$ in the Earth frame despite $L_A=2L_B$. Specifically, the two ships' speeds are related such that

$$\gamma_A \;=\; \frac{L_A}{L'_A} \;=\; 2\times\frac{L_B}{L'_B} \;=\; 2\times\gamma_B\,.$$

Finally, let's add one more datum to what we know: the velocity of the ship B was measure to be $v_B = \frac{1}{2}c$. This immediately gives us

$$\gamma_B \;=\; \frac{1}{\sqrt{1-\beta_B^2}} \;=\; \frac{1}{\sqrt{1-(\frac{1}{2})^2}} \;=\; \frac{2}{\sqrt{3}}\,. \tag{S.62}$$

Consequently,

$$\gamma_A \;=\; 2\times\gamma_B \;=\; \frac{4}{\sqrt{3}} \tag{S.63}$$

and hence

$$\beta_A \;=\; \sqrt{1-\frac{1}{\gamma_A^2}} \;=\; \sqrt{1-\frac{3}{16}} \;=\; \frac{\sqrt{13}}{4} \;\approx\; 0.90. \tag{S.64}$$

Thus, ship A travels at velocity $v_A \approx 0.90\ c \approx 2.7 \cdot 10^8$ m/s.