

Problem 1:

The key to this question is the Lorentz-invariant parameter

$$s = P_{\text{net}}^2 = (P_\pi + P_p)^2 = (P_k + P_\Sigma)^2, \quad (\text{S.1})$$

where all squares are the Lorentz squares of 4-vectors. In the center of mass frame, — or rather center of inertia frame, —  $s$  is simply the net energy square (divided by  $c^2$ ),

$$s = \frac{E_{\text{net}}^2}{c^2}. \quad (\text{S.2})$$

In order to make a  $K^+$  meson and a  $\Sigma^+$  hyperon, the net energy of the system (in the CM frame) should be larger than the net rest energy of the two particles,

$$E_{\text{net}} \geq M_K c^2 + M_\Sigma c^2, \quad (\text{S.3})$$

hence

$$s \geq (M_K + M_\Sigma)^2 c^2. \quad (\text{S.4})$$

Note that the condition (S.4) must hold in all frames of reference and not just in the CM frame because both sides of the inequality (S.4) are Lorentz invariant.

Now consider  $s$  in the pion-proton collision,

$$s = (P_\pi + P_p)^2 = (P_\pi)^2 + (P_p)^2 + 2(P_\pi \cdot P_p) = M_\pi^2 c^2 + M_p^2 c^2 + 2(P_\pi \cdot P_p). \quad (\text{S.5})$$

In the frame where the initial pion is at rest,  $P_p^\mu = (M_p c, 0, 0, 0)$ , hence

$$(P_\pi \cdot P_p) = \frac{E_\pi}{c} \times M_p c = E_\pi M_p, \quad (\text{S.6})$$

hence

$$s = M_\pi^2 c^2 + M_p^2 c^2 + 2E_\pi M_p. \quad (\text{S.7})$$

Plugging this value of  $s$  into the inequality (S.4), we arrive at

$$M_\pi^2 c^2 + M_p^2 c^2 + 2E_\pi M_p \geq (M_K + M_\Sigma)^2 c^2 \quad (\text{S.8})$$

and therefore

$$\frac{E_\pi}{c^2} \geq \frac{(M_K + M_\Sigma)^2 - M_p^2 - M_\pi^2}{2M_p}. \quad (\text{S.9})$$

Numerically,

$$\frac{(M_K + M_\Sigma)^2 - M_p^2 - M_\pi^2}{2M_p} \approx 1030 \text{ MeV}/c^2, \quad (\text{S.10})$$

So the threshold pion energy is

$$E_\pi^{\min} = 1030 \text{ MeV}. \quad (\text{S.11})$$

**Problem 2:**

(a) In components, the proper velocity is  $u^\mu = (\gamma c, \gamma d^3\text{Vol})$  while  $d\tau = dt/\gamma$ . Consequently,

$$\alpha^0 = \frac{du^0}{d\tau} = \gamma c \frac{d\gamma}{dt} \quad (\text{S.12})$$

where

$$\frac{d\gamma}{dt} = \frac{d}{dt} \frac{1}{\sqrt{1 - \boldsymbol{\beta}^2}} = \frac{\boldsymbol{\beta}}{(1 - \boldsymbol{\beta}^2)^{3/2}} \cdot \frac{d\boldsymbol{\beta}}{dt} = \gamma^3 \boldsymbol{\beta} \cdot \frac{\mathbf{a}}{c}, \quad (\text{S.13})$$

hence

$$\alpha^0 = \gamma^4 (\boldsymbol{\beta} \cdot \mathbf{a}). \quad (\text{S.14})$$

Next, the space components of  $\alpha^\mu$ :

$$\boldsymbol{\alpha} = \frac{d\mathbf{u}}{d\tau} = \gamma \frac{d(\gamma \mathbf{v})}{dt} = \gamma^2 \mathbf{a} + \gamma \frac{d\gamma}{dt} \mathbf{v} = \gamma^2 \mathbf{a} + \gamma^4 (\boldsymbol{\beta} \cdot \mathbf{a}) \boldsymbol{\beta}. \quad (\text{S.15})$$

Or in terms of the components of  $\boldsymbol{\alpha}$  parallel or perpendicular to the velocity,

$$\begin{aligned} \boldsymbol{\alpha}_\perp &= \gamma^2 \mathbf{a}_\perp, \\ \boldsymbol{\alpha}_\parallel &= \gamma^2 \mathbf{a}_\parallel + \gamma^4 \beta^2 \mathbf{a}_\parallel = \gamma^4 \mathbf{a}_\parallel. \end{aligned} \quad (\text{S.16})$$

(b) By inspection

$$\begin{aligned} u_\mu \alpha^\mu &= u^0 \alpha^0 - \mathbf{u} \cdot \boldsymbol{\alpha} = \gamma c \alpha^0 - \gamma v \alpha_\parallel \\ &= \gamma^5 c (\boldsymbol{\beta} \cdot \mathbf{a}) - \gamma^5 v a_\parallel = 0. \end{aligned} \quad (\text{S.17})$$

This is important because the proper velocity  $u^\mu$  is constrained to have  $u_\mu u^\mu \equiv c^2$ . Consequently we must have

$$\frac{d(u_\mu u^\mu)}{d\tau} = 0 \quad (\text{S.18})$$

and therefore

$$u_\mu \alpha^\mu = u_\mu \frac{du^\mu}{d\tau} = \frac{1}{2} \frac{d(u_\mu u^\mu)}{d\tau} = 0. \quad (\text{S.19})$$

(c) In light of part (a),

$$\begin{aligned} -\alpha^\mu \alpha_\mu &= -(\alpha^0)^2 + \boldsymbol{\alpha}^2 = -(\alpha^0)^2 + \boldsymbol{\alpha}_\parallel^2 + \boldsymbol{\alpha}_\perp^2 \\ &= -\gamma^8 (\boldsymbol{\beta} \cdot \mathbf{a})^2 + \gamma^8 \mathbf{a}_\parallel^2 + \gamma^4 \mathbf{a}_\perp^2 \\ &= -\gamma^8 \beta^2 \mathbf{a}_\parallel^2 + \gamma^8 \mathbf{a}_\parallel^2 + \gamma^4 \mathbf{a}_\perp^2 \\ &= \mathbf{a}_\parallel^2 (\gamma^8 (-\beta^2 + 1) = \gamma^6) + \mathbf{a}_\perp^2 \gamma^4. \end{aligned} \quad (\text{S.20})$$

Furthermore,

$$\mathbf{a}^2 - (\boldsymbol{\beta} \times \mathbf{a})^2 = \mathbf{a}_\parallel^2 + (1 - \beta^2) \mathbf{a}_\perp^2, \quad (\text{S.21})$$

hence

$$\gamma^6 [\mathbf{a}^2 - (\boldsymbol{\beta} \times \mathbf{a})^2] = \gamma^6 \mathbf{a}_\parallel^2 + \gamma^4 \mathbf{a}_\perp^2, \quad (\text{S.22})$$

and therefore

$$-\alpha^\mu \alpha_\mu = \gamma^6 \mathbf{a}_\parallel^2 + \gamma^4 \mathbf{a}_\perp^2 = \gamma^6 [\mathbf{a}^2 - (\boldsymbol{\beta} \times \mathbf{a})^2]. \quad (4)$$

Problem 3:

Consider a moving — and accelerating — charged particle in the inertial frame that at the moment  $t_0$  happens to be co-moving with the particle, so in that frame  $\mathbf{v}(t_0) = 0$  but  $\mathbf{a}(t_0) \neq 0$ . In that frame, the particle's acceleration makes it radiate EM power at the rate given by the Larmor formula

$$\frac{dU_{\text{em}}}{dt} = W_{\text{Larmor}} = \frac{Q^2 \mu_0}{6\pi c} \mathbf{a}^2. \quad (\text{S.23})$$

Moreover, the angular distribution of this EM power is proportional to  $(\mathbf{n} \times \mathbf{a})^2$ , so the power emitted into any two opposite directions  $\pm \mathbf{n}$  is exactly the same. (Or rather, it's *exactly* the same at the moment  $t_0$  when  $\mathbf{v}(t_0) = 0$ .) Consequently, the net 3-momentum of the EM radiation emitted by the particle is zero. In 4D terms, we may combine this fact with the Larmor formula for the radiation energy as

$$\frac{dP_{\text{em}}^\mu}{dt} = \frac{u^\mu}{c^2} W_{\text{Larmor}} \quad (\text{S.24})$$

and hence

$$\frac{dP_{\text{em}}^\mu}{d\tau} = \frac{u^\mu}{c} W_{\text{Larmor}} \quad (\text{S.25})$$

because in the frame we work  $dt = d\tau$ . In the same frame,

$$-\alpha^\mu \alpha_\mu = \mathbf{a}^2, \quad (\text{S.26})$$

hence

$$W_{\text{Larmor}} = \frac{Q^2 \mu_0}{6\pi c} (-\alpha^\nu \alpha_\nu). \quad (\text{S.27})$$

Altogether, this gives us

$$\frac{dP_{\text{em}}^\mu}{d\tau} = u^\mu \times \frac{Q^2 \mu_0}{6\pi c^3} (-\alpha^\nu \alpha_\nu). \quad (\text{S.28})$$

Note: we have derived this formula for a very particular inertial frame. But however we derived it, we ended up with a Lorentz-covariant formula, so once it's valid in one inertial frame it should also be valid in all other inertial frames.

Now let's spell out eq. (S.28) in 3D terms in a frame where the particle velocity does not vanish and may be comparable to the speed of light. On the LHS of eq. (S.28) for  $\mu = 0$  we have

$$\frac{dP_{\text{em}}^0}{d\tau} = \frac{\gamma}{c} \frac{dU_{\text{em}}}{dt} \quad (\text{S.29})$$

while on the RHS

$$u^\mu \times \frac{Q^2 \mu_0}{6\pi c^3} = \gamma \frac{Q^2 \mu_0}{6\pi c^2} \quad (\text{S.30})$$

and

$$(-\alpha^\nu \alpha_\nu) = \gamma^6 [\mathbf{a}^2 - (\boldsymbol{\beta} \times \mathbf{a})^2]. \quad (4)$$

Altogether, this gives us

$$\frac{\gamma}{c} \frac{dU_{\text{em}}}{dt} = \gamma \frac{Q^2 \mu_0}{6\pi c^2} \gamma^6 [\mathbf{a}^2 - (\boldsymbol{\beta} \times \mathbf{a})^2] \quad (\text{S.31})$$

and hence

$$\frac{dU_{\text{em}}}{dt} = \frac{Q^2 \mu_0}{6\pi c} \gamma^6 [\mathbf{a}^2 - (\boldsymbol{\beta} \times \mathbf{a})^2]. \quad (\text{S.32})$$

And this is precisely the Liénard–Larmor formula for the radiation by an accelerating relativistic charge.

#### Problem 4:

For the relativistic particle momentum  $\mathbf{p} = \gamma m \mathbf{v}$ , the Second Law of Newton  $d\mathbf{p}/dt = \mathbf{F}$  becomes

$$\gamma m \mathbf{a}_\perp = \mathbf{F}_\perp \quad \text{but} \quad \gamma^3 \mathbf{a}_\parallel = \mathbf{F}_\parallel. \quad (\text{S.33})$$

We may combine these 2 equations as

$$\gamma m \mathbf{a} = \mathbf{F}_\perp + \frac{1}{\gamma^2} \mathbf{F}_\parallel = \mathbf{F}_\perp + (1 - \beta^2) \mathbf{F}_\parallel = \mathbf{F} - \boldsymbol{\beta}(\boldsymbol{\beta} \cdot \mathbf{F}). \quad (\text{S.34})$$

This formula works for any kind of force acting on a relativistic particle, but let's apply it

to the electric+magnetic force

$$\mathbf{F} = Q(\mathbf{E} + \mathbf{v} \times \mathbf{B}). \quad (\text{S.35})$$

The magnetic component of this force is always  $\perp$  to the particle's velocity, thus

$$\boldsymbol{\beta} \cdot \mathbf{F} = Q\boldsymbol{\beta} \cdot \mathbf{E} + 0. \quad (\text{S.36})$$

Consequently, plugging the EM force into eq. (S.34) we get

$$\gamma m \mathbf{a} = Q(\mathbf{E} + \mathbf{v} \times \mathbf{B} - \boldsymbol{\beta}(\boldsymbol{\beta} \cdot \mathbf{E})). \quad (5)$$

Problem 5:

(a) In the rest frame  $\mathcal{S}$  of the charge  $Q_A$ , it generates the Coulomb electric field

$$\mathbf{E}(\mathbf{r}) = \frac{Q_A}{4\pi\epsilon_0} \frac{\mathbf{r}}{|\mathbf{r}|^3} \quad (\text{S.37})$$

and no magnetic field. Consequently, the force on the charge  $Q_B$  is the purely electric force

$$\mathbf{F} = Q_B \mathbf{E}(\mathbf{r}_B) = \frac{Q_A Q_B}{4\pi\epsilon_0} \frac{\mathbf{r}_B}{|\mathbf{r}_B|^3}. \quad (\text{S.38})$$

In particular, at the time  $t = 0$   $\mathbf{r}_B = b\hat{\mathbf{y}}$ , hence

$$\mathbf{F} = \frac{Q_A Q_B}{4\pi\epsilon_0} \frac{\hat{\mathbf{y}}}{b^2}. \quad (\text{S.39})$$

(b) In the rest frame  $\mathcal{S}'$  of the other charge  $Q_B$ , the charge  $Q_A$  moves at constant velocity, so it generates both electric and magnetic fields,

$$\begin{aligned}\mathbf{E}'(\mathbf{r}', t') &= \frac{Q_A}{4\pi\epsilon_0} \frac{\gamma(\mathbf{r} - \mathbf{r}_A(t'))}{[\gamma^2(x' + vt')^2 + (y + b)^2 + z^2]^{3/2}}, \\ \mathbf{B}'(\mathbf{r}', t') &= -\frac{\mathbf{v}}{c^2} \times \mathbf{E}'(\mathbf{r}', t').\end{aligned}\tag{S.40}$$

Although this time there is a non-zero magnetic field, the charge  $Q_B$  is at rest so it does not care about the magnetic field. Instead, the force on it is the purely electric field

$$\mathbf{F}'(t') = Q_B \mathbf{E}(\mathbf{r}_B = 0, t') = \frac{Q_A Q_B}{4\pi\epsilon_0} \frac{\gamma(vt' \hat{\mathbf{x}} + b \hat{\mathbf{y}})}{[\gamma^2(vt')^2 + b^2]^{3/2}}.\tag{S.41}$$

In particular, at the moment  $t' = 0$

$$\mathbf{F}' = \frac{Q_A Q_B}{4\pi\epsilon_0} \frac{\gamma \hat{\mathbf{y}}}{b^2}.\tag{S.42}$$

(c) The Lorentz transformation rules for the forces are explained in [my notes on relativistic energy and momentum](#). In particular, transforming the force  $\mathbf{F}'$  from the rest frame  $\mathcal{S}'$  of the charge  $Q_B$  to the frame  $\mathcal{S}$  where it moves at velocity  $\mathbf{v}$ , we get

$$\mathbf{F}_{\parallel} = \mathbf{F}'_{\parallel}, \quad \mathbf{F}_{\perp} = \frac{1}{\gamma} \mathbf{F}'_{\perp}.\tag{S.43}$$

For the problem at hand, the force on the  $Q_B$  is  $\perp$  to the velocity  $\mathbf{v}$ , so we should have  $\mathbf{F} = (1/\gamma)\mathbf{F}'$ . And by inspection of eq. (S.39) and (S.42) this is indeed the case:

$$\mathbf{F} = \frac{Q_A Q_B}{4\pi\epsilon_0} \frac{\hat{\mathbf{y}}}{b^2} = \frac{1}{\gamma} * \frac{Q_A Q_B}{4\pi\epsilon_0} \frac{\gamma \hat{\mathbf{y}}}{b^2} = \frac{1}{\gamma} \mathbf{F}'.\tag{S.44}$$

Problem 6:

(a) The Lorentz transform of the spacetime coordinates between the two frames is

$$t' = \gamma \left( t - \frac{v}{c^2} x \right), \quad x' = \gamma(x - vt), \quad y' = y, \quad z' = z, \quad (\text{S.45})$$

and conversely

$$t = \gamma \left( t' + \frac{v}{c^2} x' \right), \quad x = \gamma(x' + vt'), \quad y = y', \quad z = z'. \quad (\text{S.46})$$

Consequently,

$$kx - \omega t = k\gamma(x' + vt') - \omega\gamma \left( t' + \frac{vx'}{c^2} \right) = x \left( \gamma k - \frac{\omega\gamma v}{c^2} \right) - t \left( -k\gamma v + \omega\gamma \right), \quad (\text{S.47})$$

or in other words

$$kx - \omega t = k'x' - \omega't' \quad (\text{S.48})$$

for

$$\omega' = \gamma(\omega - vk), \quad k' = \gamma \left( k - \frac{v\omega}{c^2} \right). \quad (\text{S.49})$$

Next, for the same spacetime point

$$\begin{aligned} E'^x &= E^x, \\ E'^y &= \gamma(E^y - vB^z), \\ E'^z &= \gamma(E^z + vB^y), \\ B'^x &= B^x, \\ B'^y &= \gamma \left( B^y + \frac{v}{c^2} E^z \right), \\ B'^z &= \gamma \left( B^z - \frac{v}{c^2} E^y \right). \end{aligned} \quad (\text{S.50})$$

Consequently, for the wave (8) we have

$$E^x = E^z = 0, \quad B^x = B^y = 0, \quad (\text{S.51})$$

while

$$\begin{aligned}
E^y &= \gamma E_0 \cos(kx - \omega t) - \gamma v \frac{E_0}{c} \cos(kx - \omega t) \\
&= \gamma(1 - \beta) E_0 \cos(kx - \omega t), \\
B^z &= \gamma \frac{E_0}{c} \cos(kx - \omega t) - \gamma \frac{v}{c^2} E_0 \cos(kx - \omega t) \\
&= \gamma(1 - \beta) \frac{E_0}{c} \cos(kx - \omega t).
\end{aligned} \tag{S.52}$$

In light of eq. (S.48), this means

$$\mathbf{E}'(x', y', z', t') = E'_0 \cos(k'x' - \omega't') \hat{\mathbf{y}}, \quad \mathbf{B}'(x', y', z', t') = \frac{E'_0}{c} \cos(k'x' - \omega't') \hat{\mathbf{z}}, \tag{S.53}$$

for

$$E'_0 = \gamma(1 - \beta) E_0. \tag{S.54}$$

(b) The frequency and the wave number in the  $\mathcal{S}'$  are spelled out in eqs. (S.49). Now, let's assume that in the  $\mathcal{S}$  frame the wave (8) travels at the speed of light  $c$ , hence  $k = \omega/c$ . Consequently, eqs. (S.49) yield

$$\begin{aligned}
\omega' &= \gamma(\omega - vk) = \gamma\left(\omega - \frac{v\omega}{c}\right) = \gamma(1 - \beta)\omega, \\
k' &= \gamma\left(k - \frac{v\omega}{c^2}\right) = \gamma\left(\frac{\omega}{c} - \frac{v\omega}{c^2}\right) = \gamma(1 - \beta)\frac{\omega}{c}.
\end{aligned} \tag{S.55}$$

Note that  $k' = \omega'/c$ , so in the  $\mathcal{S}'$  the wave also travels at the speed of light  $c$ , in perfect agreement with the Einstein postulate.

Finally, the wavelength

$$\lambda = \frac{2\pi}{k} \implies \lambda \times \omega = 2\pi \times (\text{wave speed}). \tag{S.56}$$

The wave (8) presumably travels at the speed of light (relative to the frame  $\mathcal{S}$ ), so we should have  $k = \omega/c$  and therefore

$$\lambda = \frac{2\pi}{k} = \frac{2\pi c}{\omega} \implies \lambda \times \omega = 2\pi c. \tag{S.57}$$

But in the frame  $\mathcal{S}'$  the wave also travels at the same speed of light, so we should also have

$\lambda' \times \omega' = 2\pi c$ . And indeed, according to eqs. (S.55)  $k' = \omega/c$ , hence

$$\lambda' = \frac{2\pi}{k'} = \frac{2\pi c}{\omega'} \implies \lambda' \times \omega' = 2\pi c. \quad (\text{S.58})$$

(c) The wave amplitude in the  $\mathcal{S}'$  frame is written down in eq. (S.54),

$$E'_0 = \gamma(1 - \beta)E_0, \quad (\text{S.59})$$

or in terms of the amplitude ratio

$$\frac{E'_0}{E_0} = \gamma(1 - \beta) = \sqrt{\frac{1 - \beta}{1 + \beta}}. \quad (\text{S.60})$$

As to the wave's intensity — *i.e.*, wave power per unit of transverse area, — it obtains from the time-averaged  $x$  component of the Poynting vector. In the  $\mathcal{S}$  frame,

$$\mathbf{S} = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B} = \frac{E_0^2}{\mu_0 c} \cos^2(kx - \omega t) \hat{\mathbf{x}}, \quad (\text{S.61})$$

hence intensity

$$I = \langle S^x \rangle = \frac{E_0^2}{2\mu_0 c}. \quad (\text{S.62})$$

Likewise, in the  $\mathcal{S}'$  frame,

$$I' = \frac{E_0'^2}{2\mu_0 c}. \quad (\text{S.63})$$

Consequently, the ratio of the intensities in the two frames is

$$\frac{I'}{I} = \left( \frac{E'_0}{E_0} \right)^2 = \gamma^2(1 - \beta)^2 = \frac{1 - \beta}{1 + \beta}. \quad (\text{S.64})$$

(d) In parts (b) and (c) we saw that

$$\begin{aligned}\frac{\omega'}{\omega} &= \gamma(1 - \beta) = \sqrt{\frac{1 - \beta}{1 + \beta}}, \\ \frac{E'_0}{E_0} &= \gamma(1 - \beta) = \sqrt{\frac{1 - \beta}{1 + \beta}}, \\ \frac{I'}{I} &= \gamma^2(1 - \beta)^2 = \frac{1 - \beta}{1 + \beta}.\end{aligned}\tag{S.65}$$

Consequently, in the  $v \rightarrow c$  limit, the frequency, the amplitude, and the intensity of the wave in the moving frame all asymptote to zero. Thus, if you somehow could run beside the wave at the speed of light, you would not see any wave at all.

Problem 7:

The EM energy density, the Poynting vector, and the EM stress tensor are all components of the 4D stress-energy tensor  $\mathcal{T}^{\mu\nu}$ . In matrix form,

$$\mathcal{T}^{\mu\nu} = \begin{pmatrix} U & \frac{1}{c}S_x & \frac{1}{c}S_y & \frac{1}{c}S_z \\ \frac{1}{c}S_x & -T_{xx} & -T_{xy} & -T_{xz} \\ \frac{1}{c}S_y & -T_{yx} & -T_{yy} & -T_{yz} \\ \frac{1}{c}S_z & -T_{zx} & -T_{zy} & -T_{zz} \end{pmatrix}\tag{S.66}$$

Under Lorentz transforms, the  $\mathcal{T}^{\mu\nu}$  tensor transforms as

$$\mathcal{T}'^{\mu\nu} = L^\mu_\alpha L^\nu_\beta \mathcal{T}^{\alpha\beta},\tag{S.67}$$

and our task in this problem is to spell out this transform in components for a boost of velocity  $\mathbf{v}$  in the  $x$  direction. For this boost,

$$L^0_0 = L^x_x = \gamma, \quad L^0_x = L^x_0 = -\beta\gamma, \quad L^y_y = L^z_z = 1, \quad \text{other } L^\mu_\nu = 0,\tag{S.68}$$

hence

$$\begin{aligned}U' = \mathcal{T}'^{00} &= \gamma^2 \mathcal{T}^{00} - \beta\gamma^2 \mathcal{T}^{x0} - \beta\gamma^2 \mathcal{T}^{0x} + \beta^2 \gamma^2 \mathcal{T}^{xx} \\ &= \gamma^2 U - 2\beta\gamma^2 \frac{S^x}{c} + \beta^2 \gamma^2 (-T^{xx}),\end{aligned}\tag{S.69}$$

$$\begin{aligned}
\frac{1}{c}S'^x &= \mathcal{T}^{x0} = \gamma^2\mathcal{T}^{x0} - \beta\gamma^2\mathcal{T}^{00} - \beta\gamma^2\mathcal{T}^{xx} + \beta^2\gamma^2\mathcal{T}^{0x} \\
&= \gamma^2\frac{S^x}{c} - \beta\gamma^2U - \beta\gamma^2(-T^{xx}) + \beta^2\gamma^2\frac{S^x}{c} \\
&= \gamma^2(1 + \beta^2)\frac{S^x}{c} + \beta\gamma^2(T^{xx} - U),
\end{aligned} \tag{S.70}$$

$$\begin{aligned}
\frac{1}{c}S'^y &= \mathcal{T}^{y0} = \gamma\mathcal{T}^{y0} - \beta\gamma\mathcal{T}^{yx} \\
&= \gamma\frac{1}{c}S^y + \beta\gamma T^{yx},
\end{aligned} \tag{S.71}$$

$$\begin{aligned}
\frac{1}{c}S'^z &= \mathcal{T}^{z0} = \gamma\mathcal{T}^{z0} - \beta\gamma\mathcal{T}^{zx} \\
&= \gamma\frac{1}{c}S^z + \beta\gamma T^{zx},
\end{aligned} \tag{S.72}$$

$$\begin{aligned}
-\mathcal{T}'^{xx} &= \mathcal{T}^{xx} = \gamma^2\mathcal{T}^{xx} - \beta\gamma^2\mathcal{T}^{0x} - \beta\gamma^2\mathcal{T}^{x0} + \beta^2\gamma^2\mathcal{T}^{00} \\
&= -\gamma^2T^{xx} - 2\beta\gamma^2\frac{S^x}{c} + \beta^2\gamma^2U,
\end{aligned} \tag{S.73}$$

$$\begin{aligned}
-\mathcal{T}'^{xy} &= \mathcal{T}^{xy} = \gamma\mathcal{T}^{xy} - \beta\gamma\mathcal{T}^{0y} \\
&= -\gamma T^{xy} - \beta\gamma\frac{S^y}{c},
\end{aligned} \tag{S.74}$$

$$\begin{aligned}
-\mathcal{T}'^{xz} &= \mathcal{T}^{xz} = \gamma\mathcal{T}^{xz} - \beta\gamma\mathcal{T}^{0z} \\
&= -\gamma T^{xz} - \beta\gamma\frac{S^z}{c},
\end{aligned} \tag{S.75}$$

$$-\mathcal{T}'^{yy} = \mathcal{T}^{yy} = \mathcal{T}^{yy} = -T^{yy}, \tag{S.76}$$

$$-\mathcal{T}'^{yz} = \mathcal{T}^{yz} = \mathcal{T}^{yz} = -T^{yz}, \tag{S.77}$$

$$-\mathcal{T}'^{zz} = \mathcal{T}^{zz} = \mathcal{T}^{zz} = -T^{zz}. \tag{S.78}$$

Or in a more compact form where the indices  $a, b$  run over  $\perp$  space directions  $y, z$  only,

$$\begin{aligned}
U' &= \gamma^2 \left( U - \frac{2v}{c^2}S^x - \beta^2T^{xx} \right), \\
S'^x &= \gamma^2(1 + \beta^2)S^x + \gamma^2v(T^{xx} - U), \\
S'^a &= \gamma(S^a + vT^{xa}), \\
T'^{xx} &= \gamma^2 \left( T^{xx} + \frac{2v}{c^2}S^x - \beta^2U \right), \\
T'^{xb} &= \gamma \left( T^{xb} + \frac{v}{c^2}S^b \right), \\
T'^{ab} &= T^{ab}.
\end{aligned} \tag{S.79}$$

Problem 8:

(a) Using the Lorentz transform rules for the EM fields,

$$\begin{aligned}\mathbf{E}'_{\parallel} &= \mathbf{E}_{\parallel}, & \mathbf{E}'_{\perp} &= \gamma(\mathbf{E}_{\perp} + \mathbf{v} \times \mathbf{B}), \\ \mathbf{B}'_{\parallel} &= \mathbf{B}_{\parallel}, & \mathbf{B}'_{\perp} &= \gamma\left(\mathbf{B}_{\perp} - \frac{\mathbf{v}}{c^2} \times \mathbf{E}\right),\end{aligned}\tag{S.80}$$

we have

$$\begin{aligned}(\mathbf{E}'_{\perp})^2 - c^2(\mathbf{B}'_{\perp})^2 &= \gamma^2\left(\mathbf{E}_{\perp}^2 + 2\mathbf{E}_{\perp} \cdot (\mathbf{v} \times \mathbf{B}) + v^2\mathbf{E}_{\perp}^2\right) \\ &\quad - \gamma^2\left(c^2\mathbf{B}_{\perp}^2 - 2\mathbf{B}_{\perp} \cdot (\mathbf{v} \times \mathbf{E}_{\perp}) + \frac{v^2}{c^2}\mathbf{E}_{\perp}^2\right) \\ &= \gamma^2(1 - \beta^2)\mathbf{E}_{\perp}^2 + \gamma^2(v^2 - c^2)\mathbf{B}_{\perp}^2 \\ &\quad + 2\gamma^2\left(\mathbf{E}_{\perp} \cdot (\mathbf{v} \times \mathbf{B}_{\perp}) + \mathbf{E}_{\perp} \cdot (\mathbf{v} \times \mathbf{E}_{\perp})\right)\end{aligned}\tag{S.81}$$

where

$$\gamma^2(1 - \beta^2) = 1, \quad \gamma^2(v^2 - c^2) = -c^2\tag{S.82}$$

while

$$\mathbf{E}_{\perp} \cdot (\mathbf{v} \times \mathbf{B}_{\perp}) + \mathbf{E}_{\perp} \cdot (\mathbf{v} \times \mathbf{E}_{\perp}) = \mathbf{v} \cdot (\mathbf{B}_{\perp} \times \mathbf{E}_{\perp}) + \mathbf{v} \cdot (\mathbf{E}_{\perp} \times \mathbf{B}_{\perp}) = 0,\tag{S.83}$$

hence

$$(\mathbf{E}'_{\perp})^2 - c^2(\mathbf{B}'_{\perp})^2 = \mathbf{E}_{\perp}^2 - c^2\mathbf{B}_{\perp}^2.\tag{S.84}$$

At the same time,

$$(\mathbf{E}'_{\parallel})^2 - c^2(\mathbf{B}'_{\parallel})^2 = \mathbf{E}_{\parallel}^2 - c^2\mathbf{B}_{\parallel}^2,\tag{S.85}$$

so altogether

$$\mathbf{E}'^2 - c^2\mathbf{B}'^2 = \mathbf{E}^2 - c^2\mathbf{B}^2.\tag{S.86}$$

In other words, the combination  $\mathbf{E}^2 - c^2\mathbf{B}^2$  is invariant under Lorentz boosts, and since it's also invariant under rotations, space reflections, and time reversal, it is invariant under all Lorentz transforms.

(b) In terms of the field components  $\parallel$  or  $\perp$  to the boost velocity  $\mathbf{v}$ ,

$$\begin{aligned}\mathbf{E} \cdot \mathbf{B} &= \mathbf{E}_{\parallel} \cdot \mathbf{B}_{\parallel} + \mathbf{E}_{\perp} \cdot \mathbf{B}_{\perp}, \\ \mathbf{E}' \cdot \mathbf{B}' &= \mathbf{E}'_{\parallel} \cdot \mathbf{B}'_{\parallel} + \mathbf{E}'_{\perp} \cdot \mathbf{B}'_{\perp},\end{aligned}\tag{S.87}$$

where obviously

$$\mathbf{E}'_{\parallel} \cdot \mathbf{B}'_{\parallel} = \mathbf{E}_{\parallel} \cdot \mathbf{B}_{\parallel},\tag{S.88}$$

*cf.* eqs. (S.80), so let's verify that also

$$\mathbf{E}'_{\perp} \cdot \mathbf{B}'_{\perp} = \mathbf{E}_{\perp} \cdot \mathbf{B}_{\perp}.\tag{S.89}$$

Indeed, in light of eqs. (S.80)

$$\begin{aligned}\mathbf{E}'_{\perp} \cdot \mathbf{B}'_{\perp} &= \gamma^2(\mathbf{E}_{\perp} + \mathbf{v} \times \mathbf{B}) \cdot \left(\mathbf{B}_{\perp} - \frac{\mathbf{v}}{c^2} \times \mathbf{E}\right) \\ &= \gamma^2 \mathbf{E}_{\perp} \cdot \mathbf{B}_{\perp} + \gamma^2(\mathbf{v} \times \mathbf{B}_{\perp}) \cdot \mathbf{B}_{\perp} - \frac{\gamma^2}{c^2} \mathbf{E}_{\perp} \cdot (\mathbf{v} \times \mathbf{E}_{\perp}) \\ &\quad - \frac{\gamma^2}{c^2} (\mathbf{v} \times \mathbf{B}_{\perp}) \cdot (\mathbf{v} \times \mathbf{E}_{\perp})\end{aligned}\tag{S.90}$$

where

$$\mathbf{E}_{\perp} \cdot (\mathbf{v} \times \mathbf{E}_{\perp}) = 0, \quad (\mathbf{v} \times \mathbf{B}_{\perp}) \cdot \mathbf{B}_{\perp} = 0,\tag{S.91}$$

while

$$(\mathbf{v} \times \mathbf{B}_{\perp}) \cdot (\mathbf{v} \times \mathbf{E}_{\perp}) = v^2(\mathbf{B}_{\perp} \cdot \mathbf{E}_{\perp}).\tag{S.92}$$

Consequently,

$$\mathbf{E}'_{\perp} \cdot \mathbf{B}'_{\perp} = \gamma^2 \mathbf{E}_{\perp} \cdot \mathbf{B}_{\perp} - \gamma^2 \frac{v^2}{c^2} (\mathbf{B}_{\perp} \cdot \mathbf{E}_{\perp}) = \gamma^2(1 - \beta^2)(\mathbf{E}_{\perp} \cdot \mathbf{B}_{\perp}) = \mathbf{E}_{\perp} \cdot \mathbf{B}_{\perp},\tag{S.93}$$

and therefore

$$\mathbf{E}' \cdot \mathbf{B}' = \mathbf{E} \cdot \mathbf{B}.\tag{S.94}$$

In other words, the product  $\mathbf{E} \cdot \mathbf{B}$  is invariant under Lorentz boosts, and since it's also invariant under space rotations it is invariant under all continuous Lorentz transforms.

Alternative solution to parts (a) and (b):

The combinations  $\mathbf{E}^2 - c^2\mathbf{B}^2$  can be written in the manifestly Lorentz invariant form as

$$\begin{aligned}\mathbf{E}^2 - c^2\mathbf{B}^2 &= -\frac{c^2}{2}F_{\mu\nu}F^{\mu\nu}, \\ \mathbf{E} \cdot \mathbf{B} &= \frac{c}{8}\epsilon_{\kappa\lambda\mu\nu}F^{\kappa\lambda}F^{\mu\nu},\end{aligned}\tag{S.95}$$

where  $\epsilon_{\kappa\lambda\mu\nu}$  is the 4D [Levi–Civita tensor](#). Indeed, expanding the RH sides here in 3D terms, we have

$$\begin{aligned}F_{\mu\nu}F^{\mu\nu} &= F_{00}F^{00} + F_{0j}F^{0j} + F_{i0}F^{i0} + F_{ij}F^{ij} \\ &= 0 + \frac{(E^j)(-E^j)}{c^2} + \frac{(-E^i)(E^i)}{c^2} + (-\epsilon^{ijk}B^k)(-\epsilon^{ij\ell}B^\ell) \\ &= -\frac{2\mathbf{E}^2}{c^2} + 2\delta^{k\ell}B^k B^\ell \\ &= -2\frac{\mathbf{E}^2}{c^2} + 2\mathbf{B}^2,\end{aligned}\tag{S.96}$$

hence the first eq. (S.95). As to the second eq. (S.95), one of the 4 indices of the Levi–Civita tensor must be 0 while the other indices must be space indices, thus

$$\begin{aligned}\epsilon_{0ijk} &= +\epsilon_{3d}^{ijk}, \\ \epsilon_{i0jk} &= -\epsilon_{3d}^{ijk}, \\ \epsilon_{ij0k} &= +\epsilon_{3d}^{ijk}, \\ \epsilon_{ijk0} &= -\epsilon_{3d}^{ijk}, \\ \epsilon_{\text{other}} &= 0.\end{aligned}$$

Consequently,

$$\begin{aligned}\epsilon_{\kappa\lambda\mu\nu}F^{\kappa\lambda}F^{\mu\nu} &= \epsilon_{0ijk}F^{0i}F^{0j} + \epsilon_{i0jk}F^{i0}F^{jk} + \epsilon_{ij0k}F^{ij}F^{0k} + \epsilon_{ijk0}F^{ij}F^{k0} \\ &= \epsilon_{3d}^{ijk}\left(F^{0i}F^{jk} - F^{i0}F^{jk} + F^{ij}F^{0k} - F^{ij}F^{k0}\right) \\ &= 2\epsilon_{3d}^{ijk}\left(F^{0i}F^{jk} + F^{ij}F^{0k}\right) = 4\epsilon_{3d}^{ijk}F^{0i}F^{jk} \\ &= 4\epsilon_{3d}^{ijk}\left(\frac{-E^i}{c}\right)(-\epsilon_{3d}^{jkl}B^\ell) = \frac{4}{c}E^i B^\ell(\epsilon_{3d}^{ijk}\epsilon_{3d}^{jkl} = 2\delta^{i\ell}) \\ &= \frac{8}{c}\mathbf{E} \cdot \mathbf{B},\end{aligned}\tag{S.97}$$

and hence the second eq. (S.95).

(c) Let's use the coordinate system where the Poynting vector  $\mathbf{S} = (1/\mu_0)\mathbf{E} \times \mathbf{B}$  points in the  $\hat{\mathbf{x}}$  direction. Since the  $\mathbf{E}$  and  $\mathbf{B}$  fields are  $\perp$  to the Poynting vector, in these coordinates  $E^x = 0$  and  $B^x = 0$ . Consequently,

$$T^{xx} = \epsilon_0 E^x E^x + \frac{1}{\mu_0} B^x B^x - \delta^{xx} U = 0 - U \quad (\text{S.98})$$

while

$$T^{xy} = \epsilon_0 E^x E^y + \frac{1}{\mu_0} B^x B^y - \delta^{xy} U = 0 - 0 = 0 \quad (\text{S.99})$$

and likewise  $T^{xz} = 0$ . Thus, the matrix of the stress-energy tensor  $\mathcal{T}^{\mu\nu}$  is block-diagonal in  $(x, 0)$  and  $(y, z)$  blocks:

$$\mathcal{T}^{\mu\nu} = \begin{pmatrix} U & \frac{1}{c} S^x & 0 & 0 \\ \frac{1}{c} S^x & U & 0 & 0 \\ 0 & 0 & -T^{yy} & -T^{yz} \\ 0 & 0 & -T^{zy} & -T^{zz} \end{pmatrix}. \quad (\text{S.100})$$

Consequently, under Lorentz boosts in the  $x$  direction, the components of the upper-left block, — namely

$$-T^{xx} = U = \frac{\epsilon_0}{2} \mathbf{E}^2 + \frac{1}{2\mu_0} \mathbf{B}^2 = \frac{\epsilon_0}{2} (\mathbf{E}^2 + c^2 \mathbf{B}^2) \quad (\text{S.101})$$

and

$$\frac{1}{c} S^x = \frac{1}{c\mu_0} (\mathbf{E} \times \mathbf{B})^x = \epsilon_0 c (\mathbf{E} \times \mathbf{B})^x, \quad (\text{S.102})$$

transform into each other without mixing with any other non-zero components of the  $\mathcal{T}^{\mu\nu}$  tensor. In particular, for any boost in the  $x$  direction

$$S'^y = S'^z = 0, \quad T'^{xy} = T'^{xz} = 0. \quad (\text{S.103})$$

Now let's look for a frame where  $\mathbf{E}'$  is parallel to the  $\mathbf{B}'$  (or  $\mathbf{E}' = 0$ , or  $\mathbf{B}' = 0$ ). In any such frame  $\mathbf{E}' \times \mathbf{B}' = 0$  and hence  $\mathbf{S}' = 0$ , and since we already know that  $S'^y = S'^z = 0$

after any boost in the  $x$  direction, all we need to check is that  $S'^x = 0$ . According to the second eq. (S.79) (from problem 7),

$$S'^x = \gamma^2(1 + \beta^2)S^x + \gamma^2v(T^{xx} - U) = \gamma^2(1 + \beta^2)S^x - 2\gamma^2vU, \quad (\text{S.104})$$

so  $S'^x = 0$  if and only if

$$(1 + \beta^2)S^x = 2vU, \quad (\text{S.105})$$

or equivalently

$$\frac{\beta}{1 + \beta^2} = \frac{S^x}{2cU} = \frac{(\mathbf{E} \times \mathbf{B})^x / \mu_0}{\epsilon_0 c(\mathbf{E}^2 + c^2\mathbf{B}^2)} = \frac{c(\mathbf{E} \times \mathbf{B})^x}{\mathbf{E}^2 + c^2\mathbf{B}^2}. \quad (\text{S.106})$$

Or in vector form:  $\mathbf{E}'$  is parallel to  $\mathbf{B}'$  (or  $\mathbf{E}' = 0$ , or  $\mathbf{B}' = 0$ ) if and only if

$$\frac{\boldsymbol{\beta}}{1 + \boldsymbol{\beta}^2} = \frac{\mathbf{E} \times c\mathbf{B}}{\mathbf{E}^2 + c^2\mathbf{B}^2}. \quad (9)$$

(d) Eq. (9) has a unique solution for the vector  $\boldsymbol{\beta}$  of physical magnitude  $\beta < 1$  for any RHS as long its magnitude is strictly less than  $\frac{1}{2}$ . And indeed,

$$|\mathbf{E} \times c\mathbf{B}| \leq |\mathbf{E}| * c|\mathbf{B}| \leq \frac{1}{2}(\mathbf{E}^2 + c^2\mathbf{B}^2), \quad (\text{S.107})$$

hence

$$\left| \frac{\mathbf{E} \times c\mathbf{B}}{\mathbf{E}^2 + c^2\mathbf{B}^2} \right| \leq \frac{1}{2}. \quad (\text{S.108})$$

Moreover, this inequality becomes strict — and hence  $\beta < 1$  — unless both

$$|\mathbf{E} \times c\mathbf{B}| = |\mathbf{E}| * c|\mathbf{B}| \iff \mathbf{E} \perp \mathbf{B} \quad (\text{S.109})$$

and

$$|\mathbf{E}| * c|\mathbf{B}| = \frac{1}{2}(\mathbf{E}^2 + c^2\mathbf{B}^2) \iff |\mathbf{E}| = c|\mathbf{B}|. \quad (\text{S.110})$$

(e) Suppose one or both of the invariants (a)  $\mathbf{E}^2 - c^2\mathbf{B}^2$  and (b)  $\mathbf{E} \cdot \mathbf{B}$  does not vanish. As we have learned in parts (c) and (d), in this case there is a frame where  $\mathbf{E}' \parallel \mathbf{B}'$  (or  $\mathbf{E}' = 0$  or  $\mathbf{B}' = 0$ ). Moreover, combining the Lorentz boost (c) with a space rotation, we can make the resulting  $\mathbf{E}''$  and  $\mathbf{B}''$  fields point along any axis we like, say

$$\mathbf{E}'' = E''\hat{\mathbf{z}}, \quad \mathbf{B}'' = B''\hat{\mathbf{z}}. \quad (\text{S.111})$$

At this point, the values of the  $E''$  and  $B''$  are uniquely determined by the two invariants

$$I_1 = \mathbf{E}^2 - c^2\mathbf{B}^2 = (E'')^2 - c^2(B'')^2 \quad (\text{S.112})$$

and

$$I_2 = \mathbf{E} \cdot \mathbf{B} = E''B''. \quad (\text{S.113})$$

(Or rather, uniquely determined up to a simultaneous sign flip, which is equivalent to a  $180^\circ$  rotation around the  $x$  axis.) But any Lorentz-invariant function of the  $\mathbf{E}$  and  $\mathbf{B}$  is automatically a function of the  $E''$  and  $B''$ , so it's completely determined by the invariants  $I_1$  and  $I_2$ , and there are no more *independent* invariants.

Finally, suppose both  $I_1 = 0$  and  $I_2 = 0$ , so the  $\mathbf{E}$  and the  $\mathbf{B}$  fields are  $\perp$  to each other and their magnitudes are related as  $|\mathbf{E}| = c|\mathbf{B}|$ . For such fields there is no frame where  $\mathbf{E} \parallel \mathbf{B}$ ; instead, we can change their magnitudes to any non-zero values as long as  $E' = cB'$  and rotate their directions to anything we like as long as  $\mathbf{E}' \perp \mathbf{B}'$ . For example, let's first rotate the fields so that  $\mathbf{E}$  points in the  $y$  direction while  $\mathbf{B}$  points in the  $z$  direction. Then, a Lorentz boost in the  $x$  direction changes the electric and the magnetic magnitudes to

$$E' = \sqrt{\frac{1-\beta}{1+\beta}}E, \quad B' = \sqrt{\frac{1-\beta}{1+\beta}}B = cE'. \quad (\text{S.114})$$

This way, we may change any  $E \neq 0$  to any other  $E' \neq 0$ . Finally, after this boost, we may rotate the  $\mathbf{E}'$  and the  $\mathbf{B}'$  to point in whichever  $\perp$  directions we like.

And under to these transforms we can make, the  $\mathbf{E}$  and the  $\mathbf{B}$  fields simply do not have any invariants besides  $I_1 = \mathbf{E}^2 - c^2\mathbf{B}^2 = 0$  and  $I_2 = \mathbf{E} \cdot \mathbf{B} = 0$ .

Problem 9:

(a) The  $A^\mu = (\frac{V}{c}, \mathbf{A})$  is a Lorentz vector, so after a boost of velocity  $\mathbf{v}$ , the scalar potential becomes

$$V'(X') = \gamma(V(X) + \mathbf{v} \cdot \mathbf{A}(X)), \quad (\text{S.115})$$

which for the original potential (10) becomes

$$V'(X') = \gamma \mathbf{v} \times \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \mathbf{n}}{r^3} = \frac{\mu_0 \gamma}{4\pi} \frac{\mathbf{n} \cdot (\mathbf{v} \times \mathbf{m})}{r^3} = \frac{1}{4\pi \epsilon_0 c^2} \frac{\gamma (\mathbf{v} \times \mathbf{m}) \cdot \mathbf{r}}{|\mathbf{r}|^4}. \quad (\text{S.116})$$

In this formula

$$\mathbf{r} = \mathbf{r}' - \gamma v t' + \frac{\gamma - 1}{\beta^2} (\boldsymbol{\beta} \cdot \mathbf{r}') \boldsymbol{\beta} \quad (\text{S.117})$$

and hence

$$|\mathbf{r}|^4 = [\gamma^2 (r'_{\parallel} - vt')^2 + (\mathbf{r}'_{\perp})^2]^2. \quad (\text{S.118})$$

(b) In the non-relativistic limit of  $|\mathbf{v}| \ll c$  we have  $\gamma \approx 1$ ,

$$\mathbf{r} \approx \mathbf{r}' - \mathbf{v}t', \quad (\text{S.119})$$

hence the scalar potential (S.116) becomes

$$V'(\mathbf{r}', t') \approx \frac{1}{4\pi \epsilon_0} \frac{(\mathbf{v} \times \mathbf{m}) \cdot (\mathbf{r}' - \mathbf{v}t')}{c^2 |\mathbf{r}' - \mathbf{v}t'|^4}. \quad (\text{S.120})$$

Physically, this is the quasi-static potential of a moving electric dipole,

$$V'(\mathbf{r}', t') = \frac{1}{4\pi \epsilon_0} \frac{\mathbf{p} \cdot (\mathbf{r}' - \mathbf{r}_{\text{dipole}}(t'))}{|\mathbf{r}' - \mathbf{r}_{\text{dipole}}(t')|^4} \quad (\text{S.121})$$

for

$$\mathbf{p} = \frac{\mathbf{v} \times \mathbf{m}}{c^2}. \quad (11)$$

**Problem 10:**

(a) Let's Lorentz-boost the EM fields (12) in the  $x$  direction at velocity  $\mathbf{u}$ . Then

$$\mathbf{E}' = \gamma(\mathbf{E} + \mathbf{v} \times \mathbf{B}) = \gamma(E - uB)\hat{\mathbf{y}}. \quad (\text{S.122})$$

In particular, for

$$u = \frac{E}{B} < c \quad (\text{S.123})$$

we have  $\mathbf{E}' = 0$ , so in the moving frame  $\mathbf{S}'$  there is no electric field but only the magnetic field. Specifically,

$$\mathbf{B}' = \gamma\left(\mathbf{B} - \frac{u}{c^2} \times \mathbf{E}\right) = \gamma\left(B - \frac{uE}{c^2}\right)\hat{\mathbf{z}}, \quad (\text{S.124})$$

which for the velocity (S.123) becomes

$$\mathbf{B}' = \gamma\left(B - \frac{E^2}{Bc^2}\right)\hat{\mathbf{z}} \quad (\text{S.125})$$

where

$$\gamma = \frac{1}{\sqrt{1 - (E/Bc)^2}} = \frac{Bc}{\sqrt{B^2c^2 - E^2}}, \quad (\text{S.126})$$

hence

$$\gamma\left(B - \frac{E^2}{Bc^2}\right) = \frac{Bc}{\sqrt{B^2c^2 - E^2}} * \frac{B^2c^2 - E^2}{Bc^2} = \frac{1}{c}\sqrt{B^2c^2 - E^2} = \sqrt{B^2 - (E/c)^2}, \quad (\text{S.127})$$

and therefore

$$\mathbf{B}' = \sqrt{B^2 - (E/c)^2}\hat{\mathbf{z}}. \quad (\text{S.128})$$

(b) In the moving  $\mathcal{S}'$  frame, the particle is released with initial velocity  $\mathbf{v}_0 = -u\hat{\mathbf{x}} = -(E/B)\hat{\mathbf{x}}$  in the magnetic field (S.128). Since the initial velocity is  $\perp$  to the magnetic field, the particle moves in a circle in the  $(x', y')$  plane at constant speed  $u$ . In other words, the motion in the  $\mathcal{S}'$  frame is the cyclotron motion with velocity

$$\mathbf{v}(t') = -u \cos(\omega t')\hat{\mathbf{x}} - u \sin(\omega t')\hat{\mathbf{y}} \quad (\text{S.129})$$

where the frequency is

$$\omega = \frac{QB'}{\gamma M} = \frac{Q}{M} * \frac{B' = \sqrt{B^2 - (E/c)^2}}{\gamma = B/\sqrt{B^2 - (E/c)^2}} = \frac{Q}{M} * \frac{B^2 - (E/c)^2}{B}. \quad (\text{S.130})$$

The circle's radius is

$$R = \frac{u}{\omega} = \frac{M}{Q} * \frac{E}{B^2 - (E/c)^2}, \quad (\text{S.131})$$

so the equation of motion obtaining by integrating the velocity (S.129) over  $dt'$  is

$$x'(t') = -R \sin(\omega t'), \quad y'(t') = R \cos(\omega t') - R. \quad (\text{S.132})$$

(c) The same particle's motion in the original frame  $\mathcal{S}$  obtains by Lorentz-boosting the equations of motion (S.132) from the  $\mathcal{S}'$  frame back to  $\mathcal{S}$ . Using the  $t'$  to parametrize the particle's worldline, we have

$$\begin{aligned} t(t') &= \gamma t' + \frac{\gamma u}{c^2} x(t') = \gamma t' - \frac{\gamma u R}{c^2} \sin(\omega t') \\ &= \frac{\gamma}{\omega} (\omega t' - \beta^2 \sin(\omega t')), \end{aligned} \quad (\text{S.133})$$

$$\begin{aligned} x(t') &= \gamma x'(t') + \gamma u t' = -\gamma R \sin(\omega t') + \gamma u t' \\ &= \gamma R (\omega t' - \sin(\omega t')), \end{aligned} \quad (\text{S.134})$$

$$y(t') = y'(t') = R \cos(\omega t') - R. \quad (\text{S.135})$$

In the non-relativistic limit of  $E \ll cB$  and hence  $u \ll c$ , we may approximate eq. (S.133) as  $t \approx t'$ , and then eqs. (S.134) and (S.135) describe the *cycloid motion*

$$x(t) = R(\omega t - \sin(\omega t)), \quad y(t) = R(\cos(\omega t) - 1). \quad (\text{S.136})$$

But in the relativistic case of  $E \sim cB$  and hence  $u \sim c$ , the motion is more complicated. To describe it in terms of  $\mathcal{S}$ -frame time  $t$ , we should first solve the transcendental equation

(S.133) for the  $t'$  as a function of  $t$  and then plug it into eq. (S.134) and (S.135). The net result is a rather non-uniform motion along a stretched-out cycloid, but the only way I know how to describe it mathematically is parametric in terms of  $t'$  or  $\phi = \omega t'$ :

$$\begin{aligned}x(\phi) &= \gamma R(\phi - \sin(\phi)), \\y(\phi) &= R(\cos(\phi) - 1), \\t(\phi) &= \frac{\gamma}{\omega}(\phi - \beta^2 \sin(\phi)).\end{aligned}\tag{S.137}$$