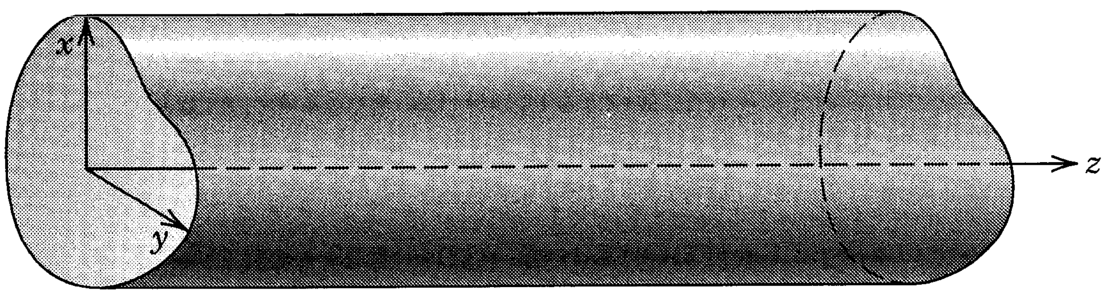
WAVEGUIDES

Waveguides are basically metal pipes carrying electromagnetic waves, usually the microwaves. In these notes, we shall start with the idealized waveguides without any dissipation of the EM energy — hence no attenuation of the waves, — and then consider the attenuation in a later section.

To avoid the attenuation, we need two things: (1) The material inside the waveguide — if any — should be linear and have have real permittivity $\epsilon(\omega)$, real permeability $\mu(\omega)$, and hence real refraction index $n(\omega)$. In these notes, I allow for general real ϵ and μ, as long as they are uniform inside the waveguide. Although in real life, most waveguides are filled with air, thus $\epsilon \approx 1$ and $\mu \approx 1$, while the rest are filled with non-magnetic dielectrics, thus $\epsilon > 1$ but $\mu \approx 1$. (2) No dissipation of EM energy by the electric currents in the waveguide's walls, so *we assume the walls to be perfect conductors.* As a consequence of perfect conductivity, the walls have negligibly small skin depth, hence the boundary conditions on the EM fields at the inner sides of the walls are

$$\mathbf{E}_{\text{tangent}} = 0 \quad \text{and} \quad \mathbf{B}_{\text{normal}} = 0. \tag{1}$$

For simplicity, let's also assume a straight-pipe geometry of the waveguide, although the pipe's cross-section can of any shape — round, rectangular, or whatever, — for example



(2)

Such waveguides have a translational symmetry in z direction, so they can carry EM waves running in z direction, the general form of such EM waves being

$$\mathbf{E}(x,y,z,t) \;=\; \vec{\mathcal{E}}(x,y)\,e^{ikz-i\omega t}, \qquad \mathbf{H}(x,y,z,t) \;=\; \vec{\mathcal{H}}(x,y)\,e^{ikz-i\omega t}. \tag{3}$$

The (x,y)–dependence of the amplitudes $\vec{\mathcal{E}}$ and $\vec{\mathcal{H}}$ here obtains from the Maxwell equations in which we replace

$$\frac{\partial}{\partial z} \;\to\; ik, \qquad \frac{\partial}{\partial t} \to\; -i\omega, \tag{4}$$

but keep the transverse derivatives $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ as they are. Such 2D residuals of Maxwell equations are best written down in 2D vector notations separating the transverse x and y components

$$\mathbf{E}_t \;=\; (E_x, E_y), \quad \mathbf{H}_t \;=\; (H_x, H_y), \quad \nabla_t \;=\; (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}) \tag{5}$$

from their longitudinal components E_z, H_z, and $\frac{\partial}{\partial z} \to ik$. Thus

$$(\nabla\cdot\mathbf{E})_{3\mathrm{d}} \;=\; \nabla_t\cdot\mathbf{E}_t \;+\; ikE_z \tag{6}$$

and likewise for the magnetic fields, so the time-independent Maxwell equations

$$\nabla\cdot\mathbf{E} \;=\; 0 \quad\text{and}\quad \nabla\cdot\mathbf{B} \;=\; 0 \quad\Longrightarrow\quad \nabla\cdot\mathbf{H} \;=\; 0 \tag{7}$$

become

$$\nabla_t\cdot\mathbf{E}_t \;+\; ikE_z \;=\; 0, \tag{M1}$$

$$\nabla_t\cdot\mathbf{H}_t \;+\; ikH_z \;=\; 0. \tag{M2}$$

As to the curl $(\nabla\times\mathbf{E})_{3\mathrm{d}}$, it has longitudinal component

$$(\nabla\times\mathbf{E})_z \;=\; (\nabla_t\times\mathbf{E}_t)_{2d} \tag{8}$$

and transverse components

$$(\nabla\times\mathbf{E})_t \;=\; ik\hat{\mathbf{z}}\times\mathbf{E}_t \;+\; \nabla_t E_z\times\hat{\mathbf{z}} \;=\; \hat{\mathbf{z}}\times\bigl(ik\mathbf{E}_t \;-\; \nabla_t E_z\bigr) \tag{9}$$

where $\hat{\mathbf{z}}$ is the unit vector in z direction, thus $\hat{\mathbf{z}}\times$(a 2d vector) rotates that 2d vector 90° to the left. Consequently, the Induction Law

$$\nabla\times\mathbf{E} \;=\; -\frac{\partial\mathbf{B}}{\partial t} \;=\; -\mu\mu_0\frac{\partial\mathbf{H}}{\partial t} \;\to\; +i\omega\mu\mu_0\,\mathbf{H} \tag{10}$$

becomes in 2D vector notations

$$(\nabla_t\times\mathbf{E}_t)_{2\mathrm{d}} \;=\; i\omega\mu\mu_0\,H_z\,, \tag{M3}$$

$$\hat{\mathbf{z}}\times\big(ik\mathbf{E}_t\,-\,\nabla_t E_z\big) \;=\; i\omega\mu\mu_0\,\mathbf{H}_t\,. \tag{M4}$$

Likewise, the Maxwell–Ampere Law

$$\nabla\times\mathbf{H} \;=\; \frac{\partial\mathbf{D}}{\partial t} \;=\; \epsilon\epsilon_0\frac{\partial\mathbf{E}}{\partial t} \;\to\; -i\omega\epsilon\epsilon_0\,\mathbf{E} \tag{11}$$

(in the absence of conduction currents inside the waveguide) becomes

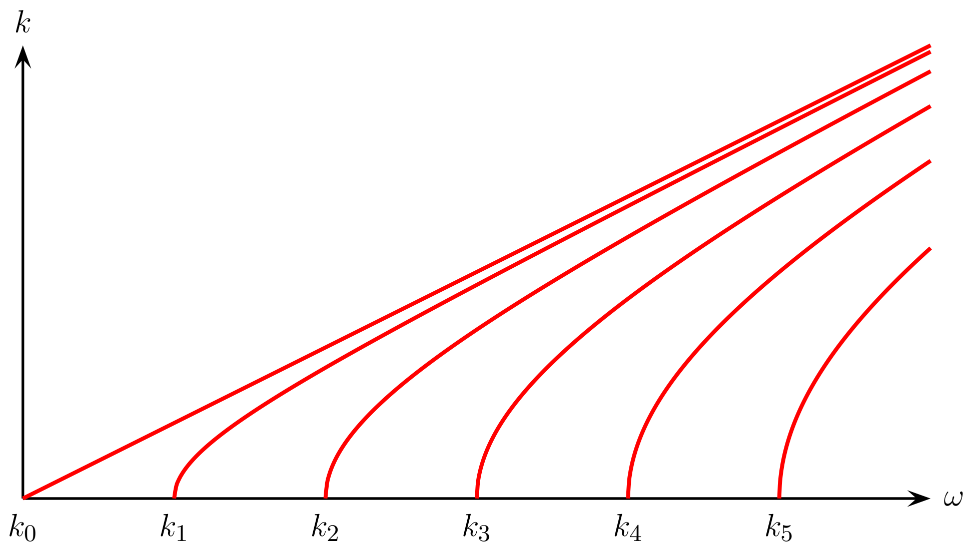
$$(\nabla_t\times\mathbf{H}_t)_{2\mathrm{d}} \;=\; -i\omega\epsilon\epsilon_0\,E_z\,, \tag{M5}$$

$$\hat{\mathbf{z}}\times\big(ik\mathbf{H}_t\,-\,\nabla_t H_z\big) \;=\; -i\omega\epsilon\epsilon_0\,\mathbf{E}_t\,. \tag{M6}$$

The general solutions of the 2D equations (M1–M6) are linear combinations of discrete *modes*, each mode having its own relation between the wave number k and the frequency ω. Specifically — as we shall see later in these notes, — for each mode#ν we have

$$k_\nu^2(\omega) \;=\; \frac{\omega^2 n^2}{c^2} \;-\; \Gamma_\nu^2 \tag{12}$$

where Γ_ν^2 is an eigenvalue of a 2D differential operator, thus



(13)

Note each mode with $\Gamma_\nu \neq 0$ has a cutoff frequency

$$\omega_{\min}(\nu) \;=\; \frac{c}{n}\,\Gamma_\nu \tag{14}$$

below which the mode cannot propagate through the waveguide. Or rather, at frequencies below the cutoff, instead of a propagating wave

$$\mathbf{E},\mathbf{H} \;\propto\; e^{ikz}e^{-i\omega t} \tag{15}$$

the mode becomes an evanescent wave

$$\mathbf{E},\mathbf{H} \;\propto\; e^{-\kappa z}e^{-i\omega t} \tag{16}$$

exponentially attenuating at the rate

$$2\kappa \;=\; 2\sqrt{\Gamma_\nu^2-(\omega n/c)^2} \;=\; \frac{2n}{c}\sqrt{\omega_{\min}^2(\nu)-\omega^2}\,. \tag{17}$$

On the other hand, at frequencies above the cutoff, the mode#ν is a propagating wave with dispersion relation (12), hence phase velocity

$$v_{\rm phase} \;=\; \frac{c}{n}\times\sqrt{\frac{\omega^2}{\omega^2-\omega_{\min}^2(\nu)}} \;>\; \frac{c}{n} \tag{18}$$

and group velocity

$$v_{\rm group} \;=\; \frac{c}{n}\times\sqrt{\frac{\omega^2-\omega_{\min}^2(\nu)}{\omega^2}} \;<\; \frac{c}{n} \tag{19}$$

(assuming $n(\omega)=\text{const}$).

Multiple wave modes propagating at different group velocities would mess up any signal transmitted by the waves. Consequently, when designing a waveguide for the microwaves carrying signals in a particular frequency range, only one mode — say mode#1 — can propagate through the waveguide while all the other modes quickly attenuate down. In terms of the cutoff frequencies (14), this means

$$\omega_{\min}(1) \;<\; \text{desired range of } \omega\text{'s} \;<\; \omega_{\min}(\nu = 2, 3, \ldots). \tag{20}$$

We shall return to this issue later in these notes once we learn how to calculate the Γ parameters — and hence the cutoff frequencies — for the modes of rectangular and circular waveguides. But before we can get there, we need to understand how the dispersion equation (12) arises in the first place.

So let's go back to the Maxwell equations in the 2D form:

$$\begin{aligned}
\nabla_t \cdot \mathbf{E}_t \;+\; ikE_z &= 0, && \text{(M1)}\\
\nabla_t \cdot \mathbf{H}_t \;+\; ikH_z &= 0, && \text{(M2)}\\
(\nabla_t \times \mathbf{E}_t)_{2\text{d}} &= i\omega\mu\mu_0\, H_z\,, && \text{(M3)}\\
\hat{\mathbf{z}} \times \big(ik\mathbf{E}_t \;-\; \nabla_t E_z\big) &= i\omega\mu\mu_0\, \mathbf{H}_t\,, && \text{(M4)}\\
(\nabla_t \times \mathbf{H}_t)_{2\text{d}} &= -i\omega\epsilon\epsilon_0\, E_z\,, && \text{(M5)}\\
\hat{\mathbf{z}} \times \big(ik\mathbf{H}_t \;-\; \nabla_t H_z\big) &= -i\omega\epsilon\epsilon_0\, \mathbf{E}_t\,. && \text{(M6)}
\end{aligned}$$

Together, eqs. (M4) and (M6) can be solved for the transverse components $\mathbf{E}_t$ and $\mathbf{H}_t$ of the EM fields in terms of the longitudinal components E_z and H_z. Indeed, let's move the transverse components to the LHS of the equations and the longitudinal components to the RHS, thus

$$\begin{aligned}
\omega\mu\mu_0\mathbf{H}_t \;-\; k\hat{\mathbf{z}} \times \mathbf{E}_t &= i\hat{\mathbf{z}} \times \nabla_t E_z\,,\\
k\hat{\mathbf{z}} \times \mathbf{H}_t \;+\; \omega\epsilon\epsilon_0\mathbf{E}_t &= -i\hat{\mathbf{z}} \times \nabla_t H_z\,.
\end{aligned} \tag{21}$$

Next, let's form linear combinations of these two equations of the form

$$\omega\epsilon\epsilon_0(\text{first}) \;+\; k\hat{\mathbf{z}} \times (\text{second}) \tag{22}$$

and

$$-k\hat{\mathbf{z}} \times (\text{first}) \;+\; \omega\mu\mu_0(\text{second}). \tag{23}$$

Using $\hat{\mathbf{z}}\times(\hat{\mathbf{z}}\times\mathbf{v}_t)=-\mathbf{v}_t$ for any transverse vector $\mathbf{v}_t$, we get

$$\begin{aligned} \omega\epsilon\epsilon_0\omega\mu\mu_0\mathbf{H}_t \;-\; \cancel{\omega\epsilon\epsilon_0 k\,\hat{\mathbf{z}}\times\mathbf{E}_t} \;+\; k^2(-\mathbf{H}_t) \;+\; \cancel{k\omega\epsilon\epsilon_0\,\hat{\mathbf{z}}\times\mathbf{E}_t} & \\ = \; i\omega\epsilon\epsilon_0\,\hat{\mathbf{z}}\times\nabla_t E_z \;+\; ik\,\nabla_t H_z\,, & \\ -\cancel{k\omega\mu\mu_0\hat{\mathbf{z}}\times\mathbf{H}_t} \;+\; k^2(-\mathbf{E}_t) \;+\; \cancel{\omega\mu\mu_0 k\,\hat{\mathbf{z}}\times\mathbf{H}_t} \;+\; \omega\mu\mu_0\omega\epsilon\epsilon_0\,\mathbf{E}_t & \\ = \; ik\,\nabla_t E_z \;-\; i\omega\mu\mu_0\,\hat{\mathbf{z}}\times\nabla_t H_z\,, & \end{aligned} \tag{24}$$

where

$$\omega\mu\mu_0\times\omega\epsilon\epsilon_0 \;=\; \omega^2\times\left(\epsilon\mu=n^2\right)\times\left(\epsilon_0\mu_0=\frac{1}{c^2}\right) \;=\; \frac{\omega^2n^2}{c^2}\,, \tag{25}$$

and therefore

$$\begin{aligned} \left(\frac{\omega^2n^2}{c^2}-k^2\right)\mathbf{H}_t \;&=\; i\omega\epsilon\epsilon_0\,\hat{\mathbf{z}}\times\nabla_t E_z \;+\; ik\nabla_t H_z\,,\\ \left(\frac{\omega^2n^2}{c^2}-k^2\right)\mathbf{E}_t \;&=\; ik\nabla_t E_z \;-\; i\omega\mu\mu_0\,\hat{\mathbf{z}}\times\nabla_t H_z\,. \end{aligned} \tag{26}$$

The physical consequences of eqs. (26) depend on whether $k=\omega n/c$ or $k\neq\omega n/c$, so let's work them out case by case.

Transverse Electromagnetic (TEM) Waves

For $k=\omega n/c$ — exactly as for a plane wave — the LH sides of eqs. (26) vanish for any transverse fields $\mathbf{E}_t$ and $\mathbf{H}_t$, so on the RHS we should have $\nabla_t E_z=0$ and $\nabla_t H_z=0$, hence $E_z(x,y)=\text{const}$ and $H_z(x,y)=\text{const}$. Moreover, at the boundary wall we should have $E_z=0$, hence $E_z\equiv 0$ throughout the waveguide. Likewise, $H_z\equiv 0$ throughout the waveguide because any $H_z(x,y)=\text{const}\neq 0$ would lead to a magnetic flux $F=\mu\mu_0 H_z\times\text{area}\neq 0$ and hence to $\text{EMF}=i\omega F\neq 0$ in the walls surrounding the waveguide. Thus, the wave in question is purely transverse, $E_z=0$ and $H_z=0$, exactly as for a plane wave, and that's why it's called a *transverse electromagnetic* (TEM) wave.

In the absence of E_z and H_z, eqs. (M4) and (M6) relate the transverse electric and magnetic fields of a TEM wave to each other exactly as in a plane wave,

$$\mathbf{H}_t(x,y) \;=\; \frac{\hat{\mathbf{z}}}{Z}\times\mathbf{E}_t(x,y), \tag{27}$$

where

$$Z \;=\; \frac{\omega\mu\mu_0}{k} \;=\; \sqrt{\frac{\mu\mu_0}{\epsilon\epsilon_0}} \tag{28}$$

is the wave impedance of the medium filling up the waveguide. Also, in the absence of E_z and H_z, Maxwell eqs. (M1) and (M3) become

$$(\nabla_t\cdot\mathbf{E}_t)_{2d} \;=\; 0 \quad\text{and}\quad (\nabla_t\times\mathbf{E}_t)_{2d} \;=\; 0, \tag{29}$$

hence

$$\mathbf{E}_t(x,y) \;=\; -\nabla_t\Phi(x,y) \tag{30}$$

for some scalar potential $\Phi(x,y)$ obeying the 2D Laplace equation

$$\triangle_{2d}\Phi \;\equiv\; \nabla_t^2\Phi \;\equiv\; \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\Phi(x,y) \;=\; 0. \tag{31}$$

The boundary condition for this Laplace equation follows from $\mathbf{E}_\perp = 0$ at the waveguide's wall. In 2D terms, this means that *at the boundary of the waveguide's cross-section in the (x,y) plane, both E_z and the tangential component of $\mathbf{E}_t$ must vanish.* For the TEM wave, $E_z = 0$ anyway, while zero tangential component of $\mathbf{E}_t = -\nabla_t\Phi$ means $\Phi = \text{const}$ along the boundary of the cross-section.

The solutions to the Laplace equation (31) subject to this Dirichlet-like boundary condition depend on the cross-section's topology. Most commonly, the waveguide is topologically a cylinder — it has an outer conducting wall of some shape but no inner conductors disconnected from the outer wall, — so its cross-section completely fills its outer boundary — which can be a circle, or a rectangle, or whatever, — but has no inner boundaries. In this case, the Laplace equation has no solutions besides the trivial $\Phi(x,y) = \text{const}$, $\mathbf{E}_t(x,y) = 0$, and *there are no TEM waves.*

On the other hand, the waveguides with both outer and inner conducting walls (disconnected from each other so that we may have $\Phi(\text{inner wall}) \neq \Phi(\text{outer wall})$) do allow for the TEM waves. For example, in a coaxial waveguide we may have

$$\Phi(\rho,\phi) \;=\; -V_0\log\rho \;+\; \text{const}, \qquad \mathbf{E}_t(\rho,\phi) \;=\; \frac{V_0}{\rho}\,\hat{\rho}\,, \tag{32}$$

and hence a non-trivial TEM wave

$$\mathbf{E}(\rho,\phi,z,t) \;=\; \frac{V_0}{\rho}\,e^{ikz-i\omega t}\,\hat{\rho}, \qquad \mathbf{H}(\rho,\phi,z,t) \;=\; \frac{V_0}{Z\rho}\,e^{ikz-i\omega t}\,\hat{\phi}. \tag{33}$$

Transverse Electric (TE) and Transverse Magnetic (TM) Waves

Now consider the wave modes with $k \neq \omega n/c$ and hence

$$\Gamma^2 \;=\; \frac{\omega^2 n^2}{c^2} \;-\; k^2 \;\neq\; 0. \tag{34}$$

For waves like these, eqs. (26) determine the transverse field components in terms of the longitudinal components,

$$\begin{aligned} \mathbf{H}_t \;&=\; \frac{i}{\Gamma^2}\Big(\omega\epsilon\epsilon_0\,\hat{\mathbf{z}}\times\nabla_t E_z \;+\; k\nabla_t H_z\Big), \\ \mathbf{E}_t \;&=\; \frac{i}{\Gamma^2}\Big(k\nabla_t E_z \;-\; \omega\mu\mu_0\,\hat{\mathbf{z}}\times\nabla_t H_z\Big). \end{aligned} \tag{35}$$

Plugging in these transverse components into the Maxwell equations (M3) and (M5) leads to the eigenvalue equations for the longitudinal fields.

$$\begin{aligned} \big(\triangle_{2d}+\Gamma^2\big)E_z(x,y) \;&=\; 0, \\ \big(\triangle_{2d}+\Gamma^2\big)H_z(x,y) \;&=\; 0. \end{aligned} \tag{36}$$

Indeed, combining Maxwell eq. (M5) with the first eq. (35), we arrive at

$$-i\omega\epsilon\epsilon_0 E_z \;=\; \big(\nabla_t\times\mathbf{H}_t\big)_{2d} \;=\; \frac{i}{\Gamma^2}\Big(\omega\epsilon\epsilon_0\nabla_t\times(\hat{\mathbf{z}}\times\nabla_t E_z) \;+\; k\nabla_t\times\nabla_t H_z\Big)_{2d} \tag{37}$$

where $\nabla_t \times \nabla_t H_z = 0$ (because $\nabla_t \times \nabla_t = 0$) while

$$\nabla_t \times (\hat{\mathbf{z}} \times \nabla_t E_z) \;=\; \hat{\mathbf{z}}\, \nabla_t^2 E_z \;-\; \nabla_t(\hat{\mathbf{z}} \cdot \nabla_t = 0) E_z \;\longrightarrow\; \nabla_t^2 E_z \quad \text{in 2d sense.} \tag{38}$$

Thus,

$$-i\omega\epsilon\epsilon_0 E_z \;=\; \frac{i}{\Gamma^2}\, \omega\epsilon\epsilon_0\, \nabla_t^2 E_z \tag{39}$$

and therefore

$$\big(\nabla_t^2 + \Gamma^2\big) E_z(x,y) \;=\; 0. \tag{40}$$

Likewise, combining the second eq. (35) with Maxwell eq. (M3) gives us

$$i\omega\mu\mu_0 H_z \;=\; \big(\nabla_t \times \mathbf{E}_t\big)_{2d} \;=\; \frac{i}{\Gamma^2}\Big(k\nabla_t \times \nabla_t E_z \;-\; \omega\mu\mu_0\, \nabla_t \times (\hat{\mathbf{z}} \times \nabla_t H_z)\Big)_{2d} \tag{41}$$

where $\nabla_t \times \nabla_t E_z \;=\; 0$ while

$$\big(\nabla_t \times (\hat{\mathbf{z}} \times \nabla_t H_z)\big)_{2d} \;=\; \nabla_t^2 H_z\,. \tag{42}$$

Thus,

$$i\omega\mu\mu_0 H_z \;=\; -\frac{I}{\Gamma^2}\, \omega\mu\mu_0\, \nabla_t^2 H_z \tag{43}$$

and consequently

$$\big(\nabla_t^2 + \Gamma^2\big) H_z(x,y) \;=\; 0. \tag{44}$$

However, while the longitudinal electric and magnetic fields obey similar-looking equations (40) and (44), they are subject to different boundary conditions (which we shall see in a moment), so in general they have different eigenvalues Γ^2. Thus, for any particular mode ν with eigenvalue Γ^2_ν, we generally have a solution of eq. (40) **or** a solution of eq. (44), but not both of them. Consequently, the wave running down the waveguide is either a *transverse magnetic (TM) wave* with $E_z \neq 0$ but $H_z = 0$, or a *transverse electric (TE) wave* with $H_z \neq 0$ but $E_z = 0$.

TRANSVERSE MAGNETIC (TM) WAVES

In a TM wave $H_z \equiv 0$, the E_z obeys the eigenstate equation

$$\big(\triangle_{2d} + \Gamma^2\big) E_z(x,y) \;=\; 0, \tag{40}$$

while the transverse electric and magnetic fields follow from the $E_z(x,y)$ — or rather, from its 2d gradient $\nabla_t E_z(x,y)$ — as

$$\mathbf{E}_t(x,y) \;=\; \frac{ik}{\Gamma^2}\,\nabla_t E_z(x,y), \qquad \mathbf{H}_t(x,y) \;=\; \frac{i\omega\epsilon\epsilon_0}{\Gamma^2}\,\hat{\mathbf{z}}\times\nabla_t E_z(x,y), \tag{45}$$

cf. eqs. (35) for $H_z \equiv 0$. In particular, these transverse fields are related to each other as

$$\mathbf{H}_t(x,y) \;=\; \frac{\hat{\mathbf{z}}}{Z}\times\mathbf{E}_t(x,y), \tag{46}$$

similar to the transverse fields in a plane wave, but for a different wave impedance

$$Z_{\rm TM} \;=\; \frac{k}{\omega\epsilon\epsilon_0} \;=\; \sqrt{\frac{\mu\mu_0}{\epsilon\epsilon_0}}\times\frac{ck}{\omega n} \;=\; Z_{\rm plane}\times\sqrt{\frac{(\omega n/c)^2-\Gamma^2}{(\omega n/c)^2}} \;<\; Z_{\rm plane}\,. \tag{47}$$

Now consider the boundary conditions for the eigenvalue equation (40). Let $\mathbf{n}$ be a unit vector $\perp$ to the waveguide's wall while $\hat{\mathbf{t}}$ is a unit vector along the wall, both $\mathbf{n}$ and $\hat{\mathbf{t}}$ lying in the (x,y) plane. In term of these vectors, the boundary conditions

$$\mathbf{E}_{\|} \;=\; 0, \qquad \mathbf{H}_{\perp} \;=\; 0 \tag{48}$$

at a perfectly conducting wall become

$$E_z \;=\; 0, \qquad \hat{\mathbf{t}}\cdot\mathbf{E}_t \;=\; 0, \qquad \mathbf{n}\cdot\mathbf{H}_t \;=\; 0. \tag{49}$$

Fortunately, the last two conditions automatically follow from the Dirichlet condition $E_z = 0$ and eqs. (45). Indeed, if $E_z = 0$ all along the boundary, then E_z has zero gradient along the boundary, $\hat{\mathbf{t}}\cdot\nabla_t E_z = 0$, and therefore

$$\begin{aligned}
\hat{\mathbf{t}}\cdot\mathbf{E}_t &= \frac{ik}{\Gamma^2}\,\hat{\mathbf{t}}\cdot\nabla_t E_z \;=\; 0,\\
\mathbf{n}\cdot\mathbf{H}_t &= \frac{i\omega\epsilon\epsilon_0}{\Gamma^2}\,\mathbf{n}\cdot(\hat{\mathbf{z}}\times\nabla_t E_z) \;=\; \frac{i\omega\epsilon\epsilon_0}{\Gamma^2}\,\nabla_t E_z\cdot(\mathbf{n}\times\hat{\mathbf{z}}=\hat{\mathbf{t}}) \;=\; 0.
\end{aligned}$$

Thus, all of the boundary conditions at a perfectly conducting wall reduce to the Dirichlet condition $E_z = 0$.

Consequently, the Γ^2 parameters of the TM waves are eigenvalues of the 2D Laplace operator (or rather $-\triangle_{2d}$) subject to the Dirichlet boundary condition,

$$-\triangle_{2d}E_z(x,y) \;=\; \Gamma^2 E_z(x,y), \qquad E_z(\text{@boundary}) \;=\; 0. \tag{50}$$

Transverse Electric (TE) Waves

In a TE wave the $E_z \equiv 0$, the H_z obeys the eigenstate equation

$$\big(\triangle_{2d}+\Gamma^2\big)H_z(x,y) \;=\; 0, \tag{40}$$

while the transverse electric and magnetic fields follow from the $H_z(x,y)$ — or rather its 2D gradient $\nabla_t H_z(x,y)$ — as

$$\mathbf{H}_t(x,y) \;=\; \frac{ik}{\Gamma^2}\,\nabla_t H_z(x,y), \qquad \mathbf{E}_t(x,y) \;=\; \frac{-i\omega\mu\mu_0}{\Gamma^2}\,\hat{\mathbf{z}}\times\nabla_t H_z(x,y), \tag{51}$$

cf. eqs. (35) for $E_z \equiv 0$. In particular, these transverse fields are related to each other as

$$\mathbf{E}_t(x,y) \;=\; -Z\hat{\mathbf{z}}\times\mathbf{H}(x,y) \quad\Longrightarrow\quad \mathbf{H}_t(x,y) \;=\; \frac{\hat{\mathbf{z}}}{Z}\times\mathbf{E}_t(x,y), \tag{52}$$

similar to the transverse fields in a plane wave or a TM wave, but for a different wave impedance

$$Z_{\rm TE} \;=\; \frac{\omega\mu\mu_0}{k} \;=\; \sqrt{\frac{\mu\mu_0}{\epsilon\epsilon_0}}\times\frac{\omega n}{ck} \;=\; Z_{\rm plane}\times\sqrt{\frac{(\omega n/c)^2}{(\omega n/c)^2-\Gamma^2}} \;>\; Z_{\rm plane}\,. \tag{53}$$

As to the boundary conditions at a perfectly conducting wall,

$$E_z \;=\; 0, \qquad \hat{\mathbf{t}}\cdot\mathbf{E}_t \;=\; 0, \qquad \mathbf{n}\cdot\mathbf{H}_t \;=\; 0, \tag{49}$$

the first condition is trivial for a TE wave while the other two conditions follow from eqs. (51) and the Neumann boundary condition

$$\mathbf{n}\cdot\nabla_t H_z \;=\; 0 \tag{54}$$

for the longitudinal magnetic field. Indeed, eqs. (54) and (51) immediately imply

$$\begin{aligned} \mathbf{n}\cdot\mathbf{H}_t \;&=\; \frac{ik}{\Gamma^2}\,\mathbf{n}\cdot\nabla_t H_z \;=\; 0,\\ \hat{\mathbf{t}}\cdot\mathbf{E}_t \;&=\; \frac{-i\omega\mu\mu_0}{\Gamma^2}\,\hat{\mathbf{t}}\cdot(\hat{\mathbf{z}}\times\nabla_t H_z) \;=\; \frac{-i\omega\mu\mu_0}{\Gamma^2}\,\nabla_t H_z\cdot(\hat{\mathbf{t}}\times\hat{\mathbf{z}}=-\mathbf{n}) \;=\; 0. \end{aligned} \tag{55}$$

Consequently, the Γ^2 parameters of the TM waves are eigenvalues of the 2D Laplace operator (or rather $-\triangle_{2d}$) subject to the Neumann boundary condition,

$$-\triangle_{2d}H_z(x,y) \;=\; \Gamma^2 H_z(x,y), \qquad \mathbf{n}\cdot\nabla_t H_z(@\text{boundary}) \;=\; 0. \tag{56}$$

Wave Energy and Power

Consider the power carried by the EM waves down the waveguide. Locally, the (time-averaged) power density is given by the Poynting vector

$$\langle \mathbf{S} \rangle \;=\; \tfrac{1}{2}\,\mathrm{Re}\big(\mathbf{E}^* \times \mathbf{H}\big) \;=\; \tfrac{1}{2}\,\mathrm{Re}\big(\mathbf{E} \times \mathbf{H}^*\big). \tag{57}$$

Let's calculate this Poynting vector for the TM and TE waves. For a TM wave

$$\mathbf{E} \;=\; E_z\hat{\mathbf{z}} \;+\; \frac{ik}{\Gamma^2}\,\nabla_t E_z\,, \qquad \mathbf{H} \;=\; \frac{i\omega\epsilon\epsilon_0}{\Gamma^2}\,\hat{\mathbf{z}} \times \nabla_t E_z\,, \tag{58}$$

hence

$$\mathbf{E}^* \times \mathbf{H} \;=\; \frac{i\omega\epsilon\epsilon_0}{\Gamma^2}\,E_z^*\hat{\mathbf{z}} \times (\hat{\mathbf{z}} \times \nabla_t E_z) \;+\; \frac{k\omega\epsilon\epsilon_0}{\Gamma^4}\,\nabla_t E_z^* \times (\hat{\mathbf{z}} \times \nabla_t E_z) \tag{59}$$

where

$$\hat{\mathbf{z}} \times (\hat{\mathbf{z}} \times \nabla_t E_z) \;=\; -\nabla_t E_z \tag{60}$$

and

$$\nabla_t E_z^* \times (\hat{\mathbf{z}} \times \nabla_t E_z) \;=\; \big(\nabla_t E_z^* \cdot \nabla_t E_z\big)\hat{\mathbf{z}}. \tag{61}$$

Altogether,

$$\mathbf{E}^* \times \mathbf{H} \;=\; -i\frac{\omega\epsilon\epsilon_0}{\Gamma^2}\,\big(E_z^*\nabla_t E_z\big) \;+\; \frac{k\omega\epsilon\epsilon_0}{\Gamma^4}\,\left|\nabla_t E_z\right|^2\,\hat{\mathbf{z}} \tag{62}$$

where the first term on RHS is imaginary and points in a transverse direction while the second term is real and points along the waveguide. The imaginary term here describes oscillations of the EM energy across the waveguide, but they do not contribute to the time-averaged power flow. Instead, the time-averaged Poynting vector stems from the second term only, thus

$$\langle \mathbf{S} \rangle \;=\; \frac{k\omega\epsilon\epsilon_0}{2\Gamma^4}\,\left|\nabla_t E_z\right|^2\,\hat{\mathbf{z}}, \tag{63}$$

so the net EM power flowing down the waveguide is

$$P_{\rm net} \;=\; \iint\limits_{\substack{\rm cross\\ \rm section}} dx\,dy\,\langle S_z\rangle \;=\; \frac{k\omega\epsilon\epsilon_0}{2\Gamma^4} \iint\limits_{\substack{\rm cross\\ \rm section}} dx\,dy\,|\nabla_t E_z|^2. \tag{64}$$

Likewise, for a TE wave

$$\mathbf{H} \;=\; H_z\hat{\mathbf{z}} \;+\; \frac{ik}{\Gamma^2}\,\nabla_t H_z\,, \qquad \mathbf{E} \;=\; -i\frac{\omega\mu\mu_0}{\Gamma^2}\,\hat{\mathbf{z}}\times\nabla_t H_z\,, \tag{65}$$

hence

$$\mathbf{E}^*\times\mathbf{H} \;=\; i\frac{\omega\mu\mu_0}{\Gamma^4}\,(\hat{\mathbf{z}}\times\nabla_t H_z^*)\times(H_z\hat{\mathbf{z}}) \;-\; \frac{k\omega\mu\mu_0}{\Gamma^4}\,(\hat{\mathbf{z}}\times\nabla_t H_z^*)\times\nabla_t H_z \tag{66}$$

where

$$(\hat{\mathbf{z}}\times\nabla_t H_z^*)\times\hat{\mathbf{z}} \;=\; +\nabla_t H_z^*\,, \tag{67}$$

$$(\hat{\mathbf{z}}\times\nabla_t H_z^*)\times\nabla_t H_z \;=\; -\big(\nabla_t H_z^*\cdot\nabla_t H_z\big)\hat{\mathbf{z}}, \tag{68}$$

and therefore

$$\mathbf{E}^*\times\mathbf{H} \;=\; i\frac{\omega\mu\mu_0}{\Gamma^4}\,H_z\nabla_t H_z^* \;+\; \frac{k\omega\mu\mu_0}{\Gamma^4}\,\left|\nabla_t H_z\right|^2\,\hat{\mathbf{z}}. \tag{69}$$

Similar to the TM case, the first term in this formula is imaginary and transverse while the second term is real and longitudinal, so only the second term contributes to the time-averaged Poynting vector. Thus,

$$\langle \mathbf{S} \rangle \;=\; \frac{k\omega\mu\mu_0}{2\Gamma^4}\,\left|\nabla_t H_z\right|^2\,\hat{\mathbf{z}} \tag{70}$$

and therefore the net EM power flowing down the waveguide

$$P_{\rm net} \;=\; \iint\limits_{\substack{\rm cross\\ \rm section}} dx\,dy\,\langle S_z\rangle \;=\; \frac{k\omega\mu\mu_0}{2\Gamma^4} \iint\limits_{\substack{\rm cross\\ \rm section}} dx\,dy\,|\nabla_t H_z|^2. \tag{71}$$

Note similar form of eqs. (64) and (71) for the two kinds of waves,

$$P_{\rm net} \;=\; \frac{k\omega}{2\Gamma^4}\iint\limits_{\substack{\rm cross\\ \rm section}} dx\,dy\Big(\epsilon\epsilon_0|\nabla_t E_z|^2 \quad\text{or}\quad \mu\mu_0|\nabla_t H_z|^2\Big). \tag{72}$$

Moreover, in both cases we have an integral of the form

$$\iint dx\,dy\,|\nabla_t\psi(x,y)|^2 \tag{73}$$

where $\psi(x,y)$ — being either $E_z(x,y)$ or $H_z(x,y)$, depending on the wave type — obeys the eigenstate equation $(\nabla_t^2+\Gamma^2)\psi(x,y)=0$. Consequently, we may simplify the integral (73) by integrating by parts:

$$\begin{aligned}\iint dx\,dy\,|\nabla_t\psi(x,y)|^2 \;&=\; \iint dx\,dy\,\nabla_t\psi^*\cdot\nabla_t\psi\\ &=\; \int\limits_{\rm boundary} d\ell\,\psi^*\nabla_t\psi\cdot\mathbf{n} \;-\; \iint dx\,dy\,\psi^*\nabla_t^2\psi.\end{aligned} \tag{74}$$

Moreover, the boundary integral here vanishes by the boundary conditions for the E_z or H_z: Dirichlet for the E_z and hence $\psi^* = E_z^* = 0$, or Neumann for the H_z and hence $\nabla_t\psi\cdot\mathbf{n} = \nabla_t H_z\cdot\mathbf{n} = 0$. Thus, we are left with

$$\iint dx\,dy\,|\nabla_t\psi(x,y)|^2 \;=\; -\iint dx\,dy\,\psi^*\nabla_t^2\psi \;=\; +\Gamma^2\iint dx\,dy\,\psi^*\psi, \tag{75}$$

where the second equality follows from eigenstate equation $\nabla_t^2\psi = -\Gamma^2\psi$. And thanks to this formula, we may simplify the integral (72) for the net EM power as

$$P_{\rm net} \;=\; \frac{k\omega}{2\Gamma^2}\iint\limits_{\substack{\rm cross\\ \rm section}} dx\,dy\Big(\epsilon\epsilon_0|E_z|^2 \quad\text{or}\quad \mu\mu_0|H_z|^2\Big). \tag{76}$$

This EM power flows along the waveguide at the group velocity

$$v_{\text{group}} \;=\; \frac{c}{\sqrt{\epsilon\mu}}\times\sqrt{\frac{\omega^2-\omega_{\text{min}}^2(\nu)}{\omega^2}} \;<\; \frac{c}{n} \tag{19}$$

To see that, let's calculate the net energy of the waves per unit length of the waveguide. The volume density of (time-averaged) EM energy us

$$\langle u\rangle \;=\; \frac{\epsilon\epsilon_0}{4}\,|\mathbf{E}|^2 \;+\; \frac{\mu\nu_0}{4}\,|\mathbf{H}|^2, \tag{77}$$

hence for a TM wave

$$\begin{aligned} \langle u\rangle \;&=\; \frac{\epsilon\epsilon_0}{4}\Bigl(|E_z|^2 \,+\, |\mathbf{E}_t|^2\Bigr) \;+\; \frac{\mu\mu_0}{4}\,|\mathbf{H}_t|^2 \\ &=\; \frac{\epsilon\epsilon_0}{4}\left(|E_z|^2 \,+\, \frac{k^2}{\Gamma^4}\,|\nabla_t E_z|^2\right) \;+\; \frac{\mu\mu_0}{4}\times\frac{(\omega\epsilon\epsilon_0)^2}{\Gamma^4}\Bigl(|\hat{\mathbf{z}}\times\nabla_t E_z|^2 \,=\, |\nabla_t E_z|^2\Bigr) \\ &=\; \frac{\epsilon\epsilon_0}{4}\,|E_z|^2 \;+\; \frac{\epsilon\epsilon_0}{4}\,\frac{|\nabla_t E_z|^2}{\Gamma^4}\times\Bigl(k^2 \,+\, \omega^2\epsilon\epsilon_0\mu\mu_0\Bigr) \end{aligned} \tag{78}$$

where

$$\begin{aligned} k^2 \,+\, \omega^2\epsilon\epsilon_0\mu\mu_0 \;&=\; k^2 \,+\, \frac{\omega^2 n^2}{c^2} \\ &=\; \frac{\omega^2 n^2}{c^2} \,-\, \Gamma^2 \,+\, \frac{\omega^2 n^2}{c^2} \\ &=\; 2\frac{\omega^2 n^2}{c^2} \,-\, \Gamma^2, \end{aligned} \tag{79}$$

therefore

$$\langle u\rangle \;=\; \frac{\epsilon\epsilon_0}{4}\left(|E_z|^2 + \frac{1}{\Gamma^4}\left(2\frac{\omega^2 n^2}{c^2} - \Gamma^2\right)\,|\nabla_t E_z|^2\right). \tag{80}$$

Likewise, for a TE wave

$$\begin{aligned} \langle u\rangle \;&=\; \frac{\mu\mu_0}{4}\Bigl(|H_z|^2 \,+\, |\mathbf{H}_t|^2\Bigr) \;+\; \frac{\epsilon\epsilon_0}{4}\,|\mathbf{E}_t|^2 \\ &=\; \frac{\mu\mu_0}{4}\left(|H_z|^2 \,+\, \frac{k^2}{\Gamma^4}\,|\nabla_t H_z|^2\right) \;+\; \frac{\epsilon\epsilon_0}{4}\times\frac{(\omega\mu\mu_0)^2}{\Gamma^4}\Bigl(|\hat{\mathbf{z}}\times\nabla_t H_z|^2 \,=\, |\nabla_t H_z|^2\Bigr) \\ &=\; \frac{\mu\mu_0}{4}\,|H_z|^2 \;+\; \frac{\mu\mu_0}{4}\,\frac{|\nabla_t H_z|^2}{\Gamma^4}\times\Bigl(k^2 \,+\, \omega^2\mu\mu_0\epsilon\epsilon_0\Bigr) \\ &=\; \frac{\mu\mu_0}{4}\left(|H_z|^2 + \frac{1}{\Gamma^4}\left(2\frac{\omega^2 n^2}{c^2} - \Gamma^2\right)\,|\nabla_t H_z|^2\right). \end{aligned} \tag{81}$$

Thus, for both kinds of waves, the time-averaged EM energy density has form

$$\langle u\rangle \;=\; \frac{1}{4}\left(|\psi|^2 \,+\, \frac{1}{\Gamma^4}\left(2\frac{\omega^2 n^2}{c^2}-\Gamma^2\right)\,|\nabla_t\psi|^2\right) \tag{82}$$

where $\psi=\sqrt{\epsilon\epsilon_0}E_z$ for a TM wave or $\psi=\sqrt{\mu\mu_0}H_z$ for a TE wave.

Integrating this volume energy density over the waveguide's cross-section, we get the line energy density

$$\begin{aligned}\frac{U}{\text{length}} &= \iint\limits_{\substack{\text{cross}\\ \text{section}}} \langle u\rangle\,dx\,dy\\ &= \frac{1}{4}\iint|\psi|^2\,dx\,dy \;+\; \frac{1}{4\Gamma^4}\left(2\frac{\omega^2n^2}{c^2}-\Gamma^2\right)\times\iint|\nabla_t\psi|^2\,dx\,dy\\ &\qquad\langle\!\langle\text{ using eq. (75) for the second integral }\rangle\!\rangle\\ &= \frac{1}{4}\iint|\psi|^2\,dx\,dy \;+\; \frac{1}{4\Gamma^4}\left(2\frac{\omega^2n^2}{c^2}-\Gamma^2\right)\times\Gamma^2\iint|\psi|^2\,dx\,dy\\ &= \left(\frac{1}{4}\iint|\psi|^2\,dx\,dy\right)\times F\end{aligned} \tag{83}$$

where

$$F \;=\; 1\,+\,\frac{1}{\Gamma^4}\left(2\frac{\omega^2n^2}{c^2}-\Gamma^2\right)\times\Gamma^2 \;=\; 1\,+\,\frac{2}{\Gamma^2}\times\frac{\omega^2n^2}{c^2}\,-\,1 \;=\; \frac{2\omega^2n^2}{c^2\Gamma^2}\,. \tag{84}$$

Thus altogether,

$$\begin{aligned}\frac{U}{\text{length}} &= \frac{\omega^2n^2}{2c^2\Gamma^2}\iint\limits_{\substack{\text{cross}\\ \text{section}}}|\psi|^2\,dx\,dy\\ &= \frac{\omega^2n^2}{2c^2\Gamma^2}\iint\limits_{\substack{\text{cross}\\ \text{section}}}\left(\epsilon\epsilon_0|E_z|^2\quad\text{or}\quad\mu\mu_0|H_z|^2\right)dx\,dy.\end{aligned} \tag{85}$$

When this EM energy flows down the waveguide with velocity v_{energy}, it transmits the net power

$$P_{\text{net}} \;=\; v_{\text{energy}} \times \frac{U}{\text{length}}\,. \tag{86}$$

Indeed, the formula for the net power

$$P_{\text{net}} \;=\; \frac{k\omega}{2\Gamma^2} \iint\limits_{\substack{\text{cross}\\ \text{section}}} dx\,dy \Big(\epsilon\epsilon_0 |E_z|^2 \quad \text{or} \quad \mu\mu_0 |H_z|^2\Big), \tag{76}$$

we have derived earlier in this section involves exactly the same integral as eq. (85) for the line energy density, the only difference being the pre-integral factors. Consequently, we may obtain the velocity of the energy flow from the ratio of those factors:

$$v_{\text{energy}} \;=\; \frac{\text{net power}}{\text{energy/length}} \;=\; \frac{k\omega}{2\Gamma^2} \Bigg/ \frac{\omega^2 n^2}{2c^2\Gamma^2} \;=\; \frac{kc^2}{\omega n^2}\,. \tag{87}$$

Hence, for

$$k^2 \;=\; \frac{\omega^2 n^2}{c^2} \;-\; \Gamma^2 \;=\; \frac{n^2}{c^2}\big(\omega^2 \;-\; \omega^2_{\min}\big) \tag{88}$$

we have

$$\frac{k}{\omega} \;=\; \frac{n}{c}\sqrt{1 \;-\; \frac{\omega^2_{\min}}{\omega^2}} \tag{89}$$

and therefore

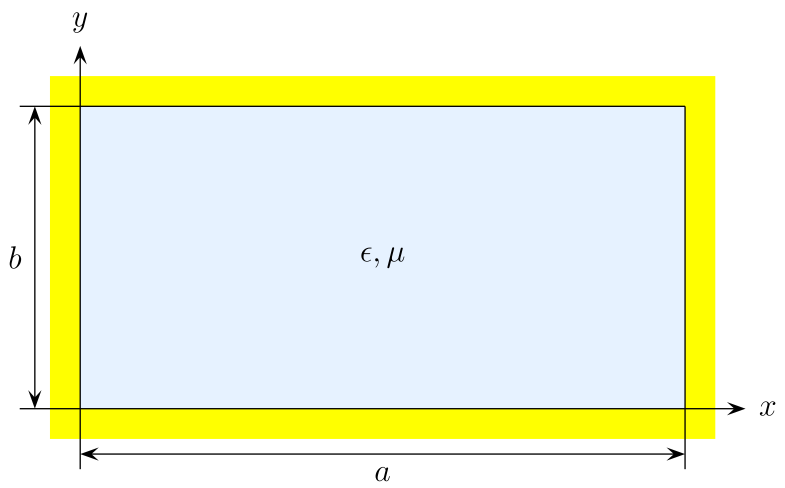
$$v_{\text{energy}} \;=\; \frac{kc^2}{\omega n^2} \;=\; \frac{c}{n}\sqrt{1 \;-\; \frac{\omega^2_{\min}}{\omega^2}} \;=\; v_{\text{group}}\,. \tag{90}$$

Thus, *the EM energy in the waveguide flows with the same velocity as the information carried by the EM waves*.

Rectangular and Circular Waveguides

Rectangular Waveguides

Many waveguides have rectangular $a \times b$ cross-sections



(91)

For this geometry, the eigenstates of the (minus) Laplacian operators obtain via the separation of variables method: We look for

$$E_z(x,y) \;=\; f(x)\times g(y) \quad\text{or}\quad H_z(x,y) \;=\; f(x)\times g(y), \tag{92}$$

hence

$$\frac{(\nabla_t^2+\Gamma^2)E_z}{E_z} \text{ or } \frac{(\nabla_t^2+\Gamma^2)H_Z}{H_z} \;=\; \frac{f''(x)}{f(x)} \;+\; \frac{g''(y)}{g(y)} \;+\; \Gamma^2 \;=\; 0, \tag{93}$$

and therefore

$$\begin{aligned} f''(x) \;&=\; -\alpha f(x) \quad \text{for a constant } \alpha, \\ g''(y) \;&=\; -\beta g(y) \quad \text{for a constant } \beta, \\ \alpha \;+\; \beta \;&=\; \Gamma^2. \end{aligned} \tag{94}$$

For the TM waves, the Dirichlet boundary conditions $E_z = 0$ on all 4 sides of the rectangle translate to

$$f(x=0) \;=\; f(x=a) \;=\; 0, \qquad g(y=0) \;=\; g(y=b) \;=\; 0, \tag{95}$$

so the solutions are

$$\begin{aligned} f(x) &= \sin\frac{m\pi x}{a} \quad \text{for an integer } m = 1, 2, 3, \ldots, \\ g(y) &= \sin\frac{n\pi y}{b} \quad \text{for an integer } n = 1, 2, 3, \ldots, \\ \Gamma^2_{m,n} &= \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2. \end{aligned} \tag{96}$$

Similarly, for the TE waves, the Neumann boundary conditions $\mathbf{n} \cdot \nabla_t H_z$ on all 4 sides translate to

$$f'(x=0) = f'(x=a) = 0, \qquad g'(y=0) = g'(y=b) = 0, \tag{97}$$

so the solutions are

$$\begin{aligned} f(x) &= \cos\frac{m\pi x}{a} \quad \text{for an integer } m = 0, 1, 2, 3, \ldots, \\ g(y) &= \cos\frac{n\pi y}{b} \quad \text{for an integer } n = 0, 1, 2, 3, \ldots, \\ \Gamma^2_{m,n} &= \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2. \end{aligned} \tag{98}$$

Note that for similar $m \geq 1$ and $n \geq 1$, the $\mathrm{TM}_{m,n}$ and the $\mathrm{TE}_{m,n}$ wave modes have exactly the same $\Gamma_{m,n}$. However, only the TE waves — but not the TM waves — may have $m = 0$ or $n = 0$. Also, for $m = n = 0$, even the TE wave does not exist: Although

$$H_z(x,y) = H_0 \cos\frac{0\pi x}{z} \times \cos\frac{0\pi y}{b} = H_0 = \text{const} \tag{99}$$

is an eigenstate of the $-\nabla_t^2$ operator with Neumann boundary conditions for the eigenvalue $\Gamma^2_{0,0} = 0$, it violates other Maxwell equations and boundary conditions. Specifically, such $H_z(x,y) = \text{const} \neq 0$ would lead to magnetic flux $F = \mu\mu_0 H_z \times ab \neq 0$ and hence EMF $= i\omega F$ in the walls surrounding the waveguide, in disagreement with the boundary condition $\mathbf{E}_{\text{tangent}} = 0$.

Altogether, the rectangular waveguide has wave modes with cutoff frequencies

$$\begin{aligned}
\omega^{\min}_{m,n} &= \frac{c}{\sqrt{\epsilon\mu}}\,\Gamma_{m,n} = \frac{\pi c}{\sqrt{\epsilon\mu}} \times \sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}} \\
&= \frac{\pi c}{\sqrt{\epsilon\mu}\,a} \times \sqrt{m^2 + n^2(a/b)^2} \\
\text{for integer } m,n &= 0,1,2,3,\ldots, \quad (m,n) \neq (0,0).
\end{aligned} \tag{100}$$

For $b < a$, the lowest cutoff frequency belongs to the $\mathrm{TE}_{1,0}$ mode, the next lowest to the $\mathrm{TE}_{2,0}$ or $\mathrm{TE}_{0,1}$ mode, depending on the aspect ratio a/b, and then we start getting both TE and TM modes. For a typical aspect ratio $a : b = 2 : 1$, the first dozen modes in the order of their cutoff frequencies are

mode	cutoff frequency in units of $\Omega_1 = \dfrac{\pi c}{\sqrt{\epsilon\mu}\,a}$
$\mathrm{TE}_{1,0}$	1.000
$\mathrm{TE}_{2,0}$ and $\mathrm{TE}_{0,1}$	2.000
$\mathrm{TE}_{1,1}$ and $\mathrm{TM}_{1,1}$	2.236
$\mathrm{TE}_{2,1}$ and $\mathrm{TM}_{2,1}$	2.828
$\mathrm{TE}_{3,0}$	3.000
$\mathrm{TE}_{3,1}$ and $\mathrm{TM}_{3,1}$	3.606
$\mathrm{TE}_{4,0}$ and $\mathrm{TE}_{0,2}$	4.000

(101)

Consequently:

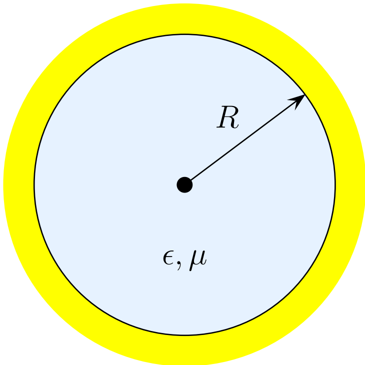
- For $\omega < \Omega_1$, all the wave modes in the waveguide are attenuating rather than propagating, so it cannot carry the waves of that frequency.
- For $\omega > \Omega_1$ but $\omega < 2\Omega_1$, the wave guide has a single propagating mode, namely $\mathrm{TE}_{1,0}$, which is good for carrying signals down the waveguide.
- For $\omega > 2\Omega_1$ the waveguide has 2 or more propagating modes with different group velocities. This is bad for carrying signals, but OK for sending a steady MW power down the waveguide.

To conclude this section, let me write down all the EM field components for the lowest wave mode $\mathrm{TE}_{1,0}$:

$$\begin{aligned} H_z(x,y,z,t) &= H_0 \times \cos\frac{\pi x}{a} \times e^{ikz-i\omega t}, \\ H_x(x,y,z,t) &= -\frac{ika}{\pi} H_0 \times \sin\frac{\pi x}{a} \times e^{ikz-i\omega t}, \\ E_y(x,y,z,t) &= +\frac{i\omega\mu\mu_0 a}{\pi} H_0 \times \sin\frac{\pi x}{a} \times e^{ikz-i\omega t}, \\ H_y &\equiv E_x \equiv E_z \equiv 0. \end{aligned} \tag{102}$$

CIRCULAR WAVEGUIDES

Now consider a waveguide with a circular cross-section



(103)

To find the eigenvalues of the (minus) Laplacian operator for this geometry, we separate the variables in the 2d polar coordinates (ρ, ϕ), thus

$$E_z(\rho,\phi) = f(\rho) \times g(\phi) \quad \text{or} \quad H_z(\rho,\phi) = f(\rho) \times g(\phi), \tag{104}$$

hence

$$0 = \frac{\rho^2}{fg} \times (\nabla_t^2 + \Gamma^2)(fg) = \frac{\rho^2 f''(\rho) + \rho f'(\rho)}{f(\rho)} + \rho^2\Gamma^2 + \frac{g''(\phi)}{g(\phi)} = 0 \tag{105}$$

and therefore

$$\begin{aligned} g''(\phi) + m^2 g(\phi) &= 0, \\ f''(\rho) + \frac{1}{\rho} f'(\rho) - \frac{m^2}{\rho^2} f(\rho) + \Gamma^2 f(\rho) &= 0 \quad \text{for the same } m^2. \end{aligned} \tag{106}$$

Solving the g equation gives us

$$g(\phi) \;=\; e^{\pm im\phi} \quad \text{for an integer } m \;=\; 0,1,2,3,\ldots, \tag{107}$$

and then the solution of the f equation which does not blow up at the center $\rho = 0$ is the Bessel function of order m,

$$f(\rho) \;=\; J_m(\Gamma\rho) \tag{108}$$

a special function defined via the integral

$$J_m(x) \;=\; \frac{1}{\pi}\int\limits_0^\pi \cos\bigl(m\tau \;-\; x\cos\tau\bigr)\,d\tau, \tag{109}$$

see these notes from an applied math class at the Brown University for more detail.

The value of the Γ in eq. (108) follows from the boundary condition at the waveguide's wall at $\rho = R$: For the TM waves we want $E_z = 0$ and hence $f(\rho = R) = 0$, while for the TE waves we want $\partial H_z/\partial\rho = 0$ and hence $f'(\rho = R) = 0$. Let $j_{m,n}$ be the n^{th} positive root of the Bessel function $J_m(x)$ while $j'_{m,n}$ is the n^{th} positive root of its derivative $dJ_m(x)/dx$. Then the TM waves have

$$\Gamma_{m,n} \;=\; \frac{j_{m,n}}{R}\,, \tag{110}$$

while the TE modes have

$$\Gamma_{m,n} \;=\; \frac{j'_{m,n}}{R}\,, \tag{111}$$

with the corresponding cutoff frequencies

$$\omega_{\rm min}(\mathrm{TM}_{m,n}) \;=\; \frac{c}{\sqrt{\mu\epsilon}\,R}\times j_{m,n}\,, \qquad \omega_{\rm min}(\mathrm{TE}_{m,n}) \;=\; \frac{c}{\sqrt{\mu\epsilon}\,R}\times j'_{m,n}\,. \tag{112}$$

Unfortunately, there are no closed formulae for the Bessel roots $j_{m,n}$ and $j'_{m,n}$, but you can find them numerically using Mathematica; here are the tables for $m \leq 5$ and $n \leq 5$ taken from the Wolfram MathWorls:

$$j_{m,n} = \tag{113}$$

n \ m	0	1	2	3	4	5
1	2.4048	3.8317	5.1356	6.3802	7.5883	8.7715
2	5.5201	7.0156	8.4172	9.7610	11.0647	12.3386
3	8.6537	10.1735	11.6198	13.0152	14.3725	15.7002
4	11.7915	13.3237	14.7960	16.2235	17.6160	18.9801
5	14.9309	16.4706	17.9598	19.4094	20.8269	22.2178

$$j'_{m,n} = \tag{114}$$

n \ m	0	1	2	3	4	5
1	3.8317	1.8412	3.0542	4.2012	5.3175	6.4156
2	7.0156	5.3314	6.7061	8.0152	9.2824	10.5199
3	10.1735	8.5363	9.9695	11.3459	12.6819	13.9872
4	13.3237	11.7060	13.1704	14.5858	15.9641	17.3128
5	16.4706	14.8636	16.3475	17.7887	19.1960	20.5755

We see that the mode with the lowest cutoff frequency is $TE_{1,1}$, the next lowest being $TM_{0,1}$. Consequently,

- For $\omega < 1.84 \times \frac{c}{\sqrt{\epsilon\mu}R}$, all the wave modes in the waveguide are attenuating rather than propagating, so it cannot carry the waves of that frequency.
- For $\omega > 1.84 \times \frac{c}{\sqrt{\epsilon\mu}R}$ but $\omega < 2.40 \times \frac{c}{\sqrt{\epsilon\mu}R}$, the wave guide has a single propagating mode, namely $TE_{1,1}$, which is good for carrying signals down the waveguide.
- For $\omega > 2.40 \times \frac{c}{\sqrt{\epsilon\mu}R}$, the waveguide has 2 or more propagating modes with different group velocities. This is bad for carrying signals, but OK for sending a steady MW power down the waveguide.

Note that for a circular waveguide, the range of frequencies for which only one mode can propagate is relatively narrow, from $\Omega_{\min}$ to about $1.36 \times \Omega_{\min}$. By comparison, the rectangular waveguides with aspect ratios $\frac{a}{b} \geq 2$ have relatively wider ranges, from $\Omega_{\min}$ to $2 \times \Omega_{\min}$.

Walls of Finite Conductivity

Thus far, we have assumed that the waveguide's walls have perfect conductivity. Now suppose the conductivity σ is finite but high enough that the skin depth

$$\delta = \sqrt{\frac{2}{\mu_{\text{metal}}\mu_0\omega\sigma}} \tag{115}$$

of the wall's material is much smaller that the waveguide's diameter or the wavelength $\lambda = 2\pi/k$. This assumption holds true for most waveguides: for example, a WR42 rectangular waveguide of size 10.7×4.32 mm carrying a wave of frequency $\omega = 2\pi \times 25$ GHz has $\lambda = 14.5$ mm, while the skin depth in copper at that frequency is only 0.4 μm.

Boundary Conditions.

At the edge of a perfectly conducting wall, the EM fields inside the waveguide obey boundary conditions

$$\mathbf{E}_{\|} = 0, \qquad H_{\perp} = 0. \tag{116}$$

For the imperfectly conducting walls, these conditions are no longer exact, but for the walls of good conductivity they remain approximately true. Specifically, the boundary conditions become

$$H_{\perp} = O(k\delta) \times H_{\|} \ll H_{\|} \tag{117}$$

and

$$E_{\|} = O(Zk\delta) \times H_{\|} \ll ZH_{\|} \sim E_{\perp}\,. \tag{118}$$

Fortunately, for the purpose of solving the eigenvalue equations for the Γ^2 of various wave modes, these boundary conditions may be approximated by the idealized conditions (116), so we do not have to redo the analysis of the previous sections.

To derive the boundary conditions (117) and (118), we start by noting that in a good conductor, the conductance current $\mathbf{J}_c = \sigma\mathbf{E}$ is much stronger than the displacement current

$\mathbf{J}_d = -i\omega\epsilon\epsilon_0\mathbf{E}$, hence

$$\nabla\times\mathbf{H} \;=\; \mathbf{J}_c \;+\; \mathbf{J}_d \;=\; (\sigma - i\omega\epsilon\epsilon_0)\mathbf{E} \;\approx\; \sigma\mathbf{E} \tag{119}$$

and therefore

$$\nabla^2\mathbf{H} \;=\; \nabla(\nabla\cdot\mathbf{H}) \;-\; \nabla\times(\nabla\times\mathbf{H}) \;=\; 0 \;-\; \sigma\nabla\times\mathbf{E} \;=\; -\sigma(i\omega\mu\mu_0)\mathbf{H} \;=\; \frac{-2i}{\delta^2}\,\mathbf{H}. \tag{120}$$

In the coordinates (ξ,η,ζ) where ξ is $\perp$ to the wall (and $\xi = 0$ at the inner surface) while η and $\zeta = z$ are tangent to the wall, the magnetic field changes with the tangent coordinates (η,ζ) on the scale $O(\lambda) \gg \delta$, so eq. (120) implies

$$\mathbf{H}(\xi,\eta,\zeta) \;\approx\; \mathbf{H}_0(\eta)\,\exp(ik\zeta)\,\exp((i-1)\xi/\delta). \tag{121}$$

The space derivatives applied to this magnetic field act as

$$\frac{\partial}{\partial\xi} \;=\; \frac{(i-1)}{\delta} \quad\text{while}\quad \frac{\partial}{\partial\zeta} \;=\; ik \quad\text{and}\quad \frac{\partial}{\partial\eta} \;=\; O(k), \tag{122}$$

so the magnetic Gauss law

$$\nabla\cdot\mathbf{H} \;=\; \frac{\partial H_\xi}{\partial\xi} \;+\; \frac{\partial H_\eta}{\partial\eta} \;+\; \frac{\partial H_\zeta}{\partial\zeta} \;=\; 0 \tag{123}$$

requires H_ξ to be much smaller that H_η or H_ζ. Specifically,

$$\frac{H_\perp}{H_\parallel} \;=\; O(k\delta) \;\ll\; 1. \tag{124}$$

As written, this relation applies inside the metal wall of the waveguide, but it can be extended inside the wave channel itself as a boundary condition at the metal's edge. Indeed, the magnetic field continues across the metal's edge as

$$\mathbf{H}_{\|}(\text{channel}) \;=\; \mathbf{H}_{\|}(\text{metal}) \quad \text{but} \quad H_{\perp}(\text{channel}) \;=\; \frac{\mu_{\text{metal}}}{\mu_{\text{channel}}}\, H_{\perp}(\text{metal}), \tag{125}$$

hence at the boundary of the wave channel

$$\frac{H_{\perp}}{H_{\|}} \;=\; \frac{\mu_{\text{metal}}}{\mu_{\text{channel}}} \times O(k\delta). \tag{126}$$

The waveguides are usually made from good electric conductors such as copper, brass, aluminum, or silver, and all these metals have $\mu \approx 1$, hence

$$\text{at the boundary of the wave channel} \quad \frac{H_{\perp}}{H_{\|}} \;=\; O(k\delta) \;\ll\; 1, \tag{127}$$

as promised in eq. (117). In principle, one can break this boundary condition by making the waveguide wall out of steel or other high-μ alloy, but nobody in his/her right mind ever does it, so let's not consider this possibility any further. Instead, we take eq. (127) as generally true, which justifies the idealized approximation $H_{\perp} \approx 0$ (in comparison to the $H_{\|}$).

Next, consider the electric field inside the metal wall. Inverting eq. (119), we have

$$\mathbf{E} \;\approx\; \frac{1}{\sigma}\, \nabla \times \mathbf{H}, \tag{128}$$

hence for the magnetic field as in eq. (122),

$$\begin{aligned} \sigma E_{\xi} \;&=\; O(k) H_{\zeta} \;-\; ikH_{\eta}\,, \\ \sigma E_{\eta} \;&=\; ikH_{\xi} \;-\; \frac{i-1}{\delta}\, H_{\zeta}\,, \\ \sigma E_{\zeta} \;&=\; \frac{i-1}{\delta}\, H_{\eta} \;-\; O(k) H_{\xi}\,. \end{aligned} \tag{129}$$

In magnitude, the tangent E_{η}, E_{ζ} components of this electric field are

$$\sigma E_{\|} \;=\; O\!\left(\frac{H_{\|}}{\delta}\right) \;+\; O(k \times H_{\perp}) \;=\; O\!\left(\frac{H_{\|}}{\delta}\right), \tag{130}$$

hence

$$\frac{E_{\|}}{H_{\|}} \sim \frac{1}{\delta\sigma} \sim \delta k Z \ll Z \tag{131}$$

where Z is the wave impedance of the waveguide, and the second relation stems from

$$\frac{1}{\delta\sigma} \Big/ \delta k Z = \left(\frac{1}{\delta^2} = \frac{\omega\mu_{\text{metal}}\mu_0\sigma}{2}\right) \times \frac{1}{\sigma k Z} = \frac{\omega\mu_{\text{metal}}\mu_0}{2kZ} = \frac{\mu_{\text{metal}}}{2\mu_{\text{channel}}} \sim 1. \tag{132}$$

The relation (131) holds inside the metal wall, but since the tangent components $E_{\|}$ and $H_{\|}$ of both electric and magnetic fields are continuous across the metal's surface, we see that the boundary values of the tangent fields inside the wave channels also obey

$$\frac{E_{\|}}{H_{\|}} \sim \delta k Z \ll Z, \tag{133}$$

exactly as promised in eq. (118).

Finally, the normal component $E_{\perp}$ of electric field is discontinuous across the metal's boundary, so it is not subject to any boundary conditions stemming from the skin effect. However, everywhere inside the waveguide $\mathbf{E} \sim Z\mathbf{H}$, so the dominant components of the electric and the magnetic fields at the boundary should also have a similar relation $E_{\perp} \sim Z H_{\|}$. Comparing this relation to eq. (133) for the $E_{\|}$, we immediately see that

$$\frac{E_{\|}}{E_{\perp}} \sim \frac{\delta k Z H_{\|}}{Z H_{\|}} = \delta k \ll 1, \tag{134}$$

which justifies the $E_{\|} \approx 0$ approximation.

Attenuation

Another consequence of the imperfectly conducting waveguide walls is the wave energy loss to the Ohmic resistance and hence slow attenuation of the wave power,

$$\text{power} \propto e^{-\alpha z} \tag{135}$$

for

$$\alpha = \frac{(\text{power loss})/\text{length}}{(\text{net power})}. \tag{136}$$

The industry-standard copper waveguides for centimeter-range microwaves) have attenuation rates ranging from 0.11 dB/m (1/40 m) for WR-90 at 10 GHz to 2.7 dB/m (1/60 cm) for WR-10 at 90 GHz.

So let's calculate the power loss due to Ohmic resistance in the walls and hence the attenuation rate (136). In my notes on the skin effect, I have calculated the power loss in terms of the tangent electric field $\mathbf{E}_{\parallel}$ on the metal's surface, but that is not very useful for the current purposes since in the previous sections of the current notes we have solved the wave equations using the $\mathbf{E}_{\parallel} \approx 0$ approximation. Instead, let's calculate the power loss in terms of the tangent magnetic field $\mathbf{H}_{\parallel}$ which does not even approximately vanish at the metal's surface.

Given the tangent magnetic field just outside the metal's surface, — and hence also just inside the surface, — we can find the conduction current inside the metal as

$$\mathbf{J} = \nabla\times\mathbf{H} \approx \frac{i-1}{\delta}\,\hat{\xi}\times\mathbf{H}_{\parallel} = \frac{i-1}{\delta}\,e^{(i-1)\xi/\delta}\,\hat{\xi}\times\mathbf{H}_{\parallel}^{\rm surface}, \tag{137}$$

hence density of the dissipated power

$$\frac{\text{power}}{\text{volume}} = \frac{|\mathbf{J}|^2}{2\sigma} = \frac{1}{\sigma\delta^2}\,e^{-2\xi/\delta}\left|\mathbf{H}_{\parallel}^{\rm surface}\right|^2. \tag{138}$$

Integrating this power density over the metal's depth ξ yields

$$\int\limits_0^{\text{large}\times\delta} e^{-2\xi/\delta}\,d\xi \approx \int\limits_0^{\infty} e^{-2\xi/\delta}\,d\xi = \frac{\delta}{2}, \tag{139}$$

hence

$$\frac{\text{power loss}}{\text{wall area}} = \frac{1}{2\sigma\delta}\left|\mathbf{H}_{\parallel}^{\rm surface}\right|^2 \tag{140}$$

and therefore

$$\frac{\text{power loss}}{\text{waveguide length}} = \frac{1}{2\delta\sigma}\oint\limits_{\mathcal{C}} d\eta\,\big(|H_\eta|^2 + |H_\zeta|^2\big), \tag{141}$$

where the integral is over the boundary $\mathcal{C}$ of the waveguide's cross-section and η is the coordinate along that boundary.

In the Appendix to these notes, I calculate the integral (141) for all the $\mathrm{TM}_{\mathrm{m,n}}$ and $\mathrm{TE}_{\mathrm{m,n}}$ waves in a rectangular waveguide. But for the moment, let's not assume any particular cross-sectional geometry or a particular TM or TE mode, so instead of calculating the integral (141) let's simply estimate its value up to unknown $O(1)$ numeric constants which do not depend on the wave's amplitude or frequency. For a TM wave of amplitude E_0 — that is,

$$E_z(x,y) \sim E_0, \qquad H_z(x,y) \equiv 0, \tag{142}$$

we generally have

$$\nabla_t E_z(x,y) \sim \Gamma E_0, \tag{143}$$

$$\mathbf{H}_t(x,t) = -\frac{i\omega\epsilon\epsilon_0}{\Gamma^2}\,\hat{\mathbf{z}}\times\nabla_t E_z(x,y) \sim \frac{\omega\epsilon\epsilon_0}{\Gamma^2}\times\Gamma E_0\,, \tag{144}$$

hence at the boundary

$$H_\zeta = 0, \qquad H_\eta \sim \frac{\omega\epsilon\epsilon_0}{\Gamma}\,E_0\,, \tag{145}$$

and therefore

$$\oint\limits_{\mathcal{C}} d\eta\left(|H_\eta|^2 + |H_\zeta|^2\right) \sim \left(\frac{\omega\epsilon\epsilon_0}{\Gamma}\right)^2\times|E_0|^2\times\text{perimeter}. \tag{146}$$

For a waveguide of a particular geometry and characteristic width a — such as the larger width of a rectangular waveguide, or a diameter of a circular waveguide, — the integral (146) becomes

$$\oint\limits_{\mathcal{C}} d\eta\left(|H_\eta|^2 + |H_\zeta|^2\right) = A(\text{mode})\,\times a\left(\frac{\omega\epsilon\epsilon_0}{\Gamma}\right)^2\times|E_0|^2 \tag{147}$$

where A(mode) is an $O(1)$ dimensionless number depending on a particular TM wave mode, and also on the waveguide's cross-sectional shape, but independent on the wave's frequency or amplitude. Consequently,

$$\frac{\text{(power loss)}}{\text{length}} \;=\; \frac{Aa}{2\sigma\delta}\left(\frac{\omega\epsilon\epsilon_0}{\Gamma}\right)^2 \times |E_0|^2. \tag{148}$$

At the same time, the net power transmitted down the waveguide follows from the integral

$$\iint\limits_{\substack{\text{cross}\\\text{section}}} |E_z|^2\,dx\,dy \;\sim\; |E_0|^2 \times \text{area}, \tag{149}$$

so we may parametrize this integral as

$$\iint\limits_{\substack{\text{cross}\\\text{section}}} |E_z|^2\,dx\,dy \;=\; B(\text{mode}) \times a^2|E_0|^2 \tag{150}$$

where $B(\text{mode})$ is another $O(1)$ dimensionless number depending on a specific TM mode and a specific waveguide geometry. In terms of this number, the net power transmitted down the waveguide is

$$\begin{aligned} P_{\text{net}} &= \frac{k\omega\epsilon\epsilon_0}{2\Gamma^2} \times \iint\limits_{\substack{\text{cross}\\\text{section}}} |E_z|^2\,dx\,dy \\ &= \frac{k\omega\epsilon\epsilon_0}{2\Gamma^2} \times Ba^2 \times |E_0|^2. \end{aligned} \tag{151}$$

Comparing this net power to the power loss (148) per unit length, we find the attenuation rate

$$\begin{aligned} \alpha &= \frac{\text{(power loss)/length}}{\text{(net power)}} \\ &= \frac{(Aa/2\sigma\delta)\,(\omega\epsilon\epsilon_0/\Gamma)^2 \times |E_0|^2}{(Ba^2/2)\,(k\omega\epsilon\epsilon_0/\Gamma^2) \times |E_0|^2} \\ &= \frac{A/B}{a\sigma\delta} \times \frac{\omega\epsilon\epsilon_0}{k}\,. \end{aligned} \tag{152}$$

Nest consider a TE wave of amplitude H_0, that is

$$E_z(x,y) \;\equiv\; 0, \qquad H_z(x,y) \;\sim\; H_0\,, \tag{153}$$

hence

$$\nabla_t H_z(x,y) \;\sim\; \Gamma H_0, \tag{154}$$

and therefore

$$\mathbf{H}_t(x,y) \;=\; \frac{ik}{\Gamma^2}\,\nabla_t H_z(x,y) \;\sim\; \frac{k}{\Gamma}\,H_0\,. \tag{155}$$

In particular, at the boundary

$$H_\zeta \;=\; H_z \;\sim\; H_0, \qquad H_\eta \;\sim\; \frac{k}{\Gamma}\,H_0\,, \tag{156}$$

and therefore

$$\oint\limits_{\mathcal{C}} d\eta\,|H_\zeta|^2 \;\sim\; (\text{perimeter})\times|H_0|^2, \qquad \oint\limits_{\mathcal{C}} d\eta\,|H_\eta|^2 \;\sim\; (\text{perimeter})\times\frac{k^2}{\Gamma^2}\times|H_0|^2, \tag{157}$$

which we may parametrize as

$$\begin{aligned} \oint\limits_{\mathrm{C}} d\eta\,|H_\zeta|^2 \;&=\; C(\text{mode})\times a\times|H_0|^2, \\ \oint\limits_{\mathrm{C}} d\eta\,|H_\eta|^2 \;&=\; D(\text{mode})\times a\times\frac{k^2}{\Gamma^2}\times|H_0|^2, \end{aligned} \tag{158}$$

for some mode-dependent — but frequency-independent and amplitude-independent — $O(1)$ numbers $C(\text{mode})$ and $D(\text{mode})$. In terms of these parameters, the EM power loss per unit length becomes

$$\begin{aligned} \frac{(\text{power loss})}{\text{length}} \;&=\; \frac{1}{2\sigma\delta}\oint\limits_{\mathcal{C}} d\eta\left(|H_\eta|^2\,+\,|H_\zeta|^2\right) \\ &=\; \frac{a}{2\sigma\delta}\left(C(\text{mode})\,+\,\frac{k^2}{\Gamma^2}\times D(\text{mode})\right)\times|H_0|^2. \end{aligned} \tag{159}$$

At the same time,

$$\iint\limits_{\substack{\text{cross}\\\text{section}}} |H_z|^2\,dx\,dy \;=\; E(\text{mode})\times a^2\times|H_0|^2 \tag{160}$$

for a yet another mode-dependent dimensionless $O(1)$ number E, hence the net power of the wave is related to its amplitude as

$$P_{\rm net} \;=\; \frac{k\omega\mu\mu_0}{2\Gamma^2}\times E(\text{mode})\times a^2\times|H_0|^2. \tag{161}$$

Consequently, comparing the power loss rate (159) to this net power, we get the attenuation rate

$$\begin{aligned}\alpha &= \frac{(\text{power loss})/\text{length}}{(\text{net power})}\\ &= \frac{(a/2\sigma\delta)\times\bigl(C+D(k/\Gamma)^2\bigr)\times|H_0|^2}{(Ea^2/2)\,(k\omega\mu\mu_0/\Gamma^2)\times|H_0|^2}\\ &= \frac{1}{a\sigma\delta}\times\frac{C\Gamma^2+Dk^2}{Ek\omega\mu\mu_0}\,.\end{aligned} \tag{162}$$

Now let's bring eq. (162) and (152) for the attenuation rate to a common form

$$\alpha \;=\; \frac{F(\text{mode})}{aZ_{pw}\sigma\delta}\times\frac{\omega^2+G(\text{mode})\Omega^2}{\omega\sqrt{\omega^2-\Omega^2}} \tag{163}$$

where $\Omega=\omega_{\rm min}=(c/n)\Gamma$ is the cutoff frequency for the mode in question, while

$$Z_{pw} \;=\; \sqrt{\frac{\mu\mu_0}{\epsilon\epsilon_0}} \tag{164}$$

is the wave impedance of a plane wave in the material filling the waveguide's channel; for the evacuated or air-filled waveguides, $Z_{pw}\approx 377\ \Omega$. To bring eq. (162) for the TE waves to the form (163), we use

$$k \;=\; \sqrt{\left(\frac{\omega n}{c}\right)^2-\Gamma^2} \;=\; \frac{n}{c}\sqrt{\omega^2-\Omega^2}\,, \tag{165}$$

hence

$$C\Gamma^2\;+\;Dk^2 \;=\; \frac{n^2}{c^2}\bigl(C\Omega^2\;+\;D(\omega^2-\Omega^2)\bigr) \;=\; \frac{Dn^2}{c^2}\left(\omega^2\;+\;\left(\frac{C}{D}-1\right)\Omega^2\right) \tag{166}$$

while

$$Ek\omega\mu\mu_0 \;=\; \frac{En\mu\mu_0}{c}\times\omega\sqrt{\omega^2-\Omega^2}, \tag{167}$$

and therefore

$$\alpha \;=\; \frac{D/E}{a\sigma\delta}\times\frac{(n/c)^2}{(n/c)\mu\mu_0}\times\frac{\omega^2\,+\,(\frac{C}{D}-1)\Omega^2}{\omega\sqrt{\omega^2-\Omega^2}}\,. \tag{168}$$

The second factor here amounts to

$$\frac{(n/c)^2}{(n/c)\mu\mu_0}\;=\;\frac{n=\sqrt{\epsilon\mu}}{c\mu\mu_0}\;=\;\frac{1}{Z_{pw}}\,, \tag{169}$$

so eq. (168) for the attenuation rate indeed takes form

$$\alpha \;=\; \frac{F(\text{mode})}{aZ_{pw}\sigma\delta}\times\frac{\omega^2+G(\text{mode})\Omega^2}{\omega\sqrt{\omega^2-\Omega^2}} \tag{163}$$

where we identify

$$F(\text{TE mode}) \;=\; \frac{D(\text{mode})}{E(\text{mode})}\,,\qquad G(\text{TE mode}) \;=\; \frac{C(\text{mode})}{D(\text{mode})}\;-\;1. \tag{170}$$

Similarly, for the TM waves we have

$$\alpha \;=\; \frac{A/B}{a\sigma\delta}\times\frac{\omega\epsilon\epsilon_0}{k} \tag{152}$$

where

$$\frac{\omega\epsilon\epsilon_0}{k}\;=\;\frac{\epsilon\epsilon_0}{n/c}\times\frac{\omega}{\sqrt{\omega^2-\Omega^2}}\;=\;\frac{1}{Z_{pw}}\times\frac{\omega^2+0}{\omega\sqrt{\omega^2-\Omega^2}}\,, \tag{171}$$

hence

$$\alpha \;=\; \frac{A/B}{aZ_{pw}\sigma\delta}\times\frac{\omega^2+0}{\omega\sqrt{\omega^2-\Omega^2}}\,, \tag{172}$$

which also has form

$$\alpha \;=\; \frac{F(\text{mode})}{aZ_{pw}\sigma\delta}\times\frac{\omega^2+G(\text{mode})\Omega^2}{\omega\sqrt{\omega^2-\Omega^2}} \tag{163}$$

where we identify

$$F(\text{TM mode}) \;=\; \frac{A(\text{mode})}{B(\text{mode})}\,,\qquad G(\text{TM mode}) \;=\; 0. \tag{173}$$

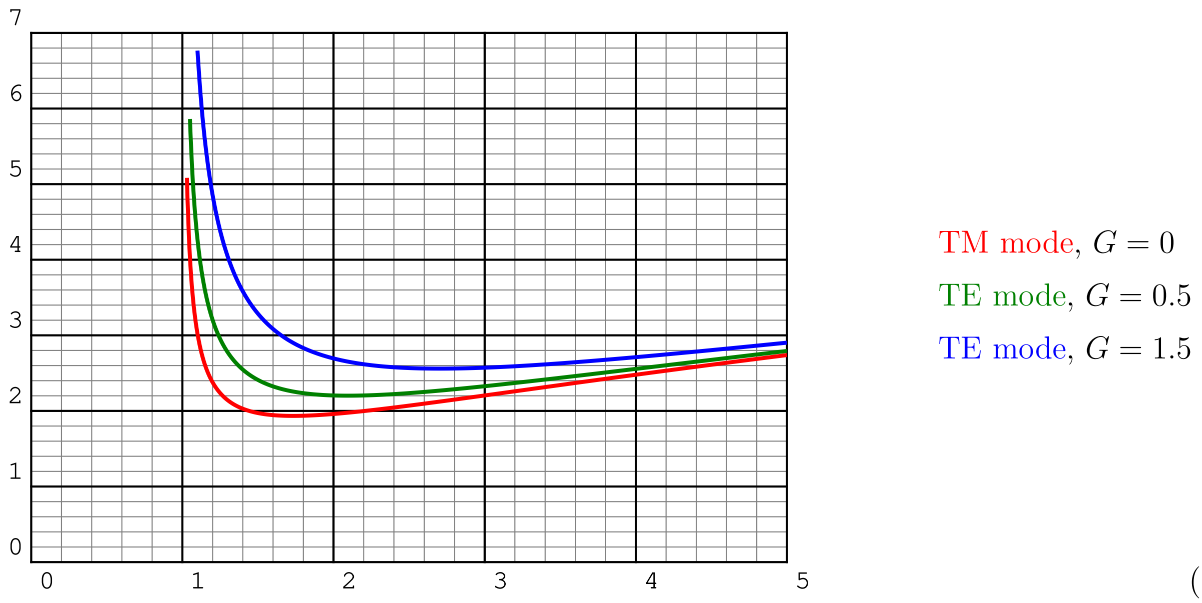
Now consider the frequency dependence of the attenuation rate. Beside the explicitly ω–dependent second factor in eq. (163), the skin depth δ (in the denominator of the first factor) depends on frequency as $\delta \propto \omega^{-1/2}$, thus

$$\sigma(\omega) \;=\; \sigma(\Omega) \times \sqrt{\frac{\Omega}{\omega}} \tag{174}$$

and hence

$$\alpha(\omega) \;=\; \frac{F(\text{mode})}{aZ_{pw}\sigma\delta(\Omega)} \times \frac{\omega^2 + G(\text{mode})\Omega^2}{\sqrt{\Omega\omega(\omega^2 - \Omega^2)}}\,. \tag{175}$$

In this form, the first factor depends on the waveguide's design and on the specific mode but does not depend on the wave's frequency, while the second factor depends on the frequency or rather on the ratio ω/Ω of the wave's frequency to the cutoff frequency for the mode in question. Graphically, α as a function of ω/Ω behaves as



(176)

Note that for all the modes, we have strong attenuation for frequencies just above the cutoff frequency Ω. Physically, this is caused by the slow velocity of the energy flow $v_{\text{energy}} = (c/n)\sqrt{1 - (\Omega/\omega)^2}$, so the energy does not move very far while it's dissipated by the conduction currents. At higher frequencies, the attenuation becomes weaker, reaches a minimum, and then starts slowly growing with the frequency due to shrinking skin depth. The optimal frequency which minimizes the attenuation rate depends on the G parameter of the mode in question, specifically

$$\left(\frac{\omega_{\text{optimal}}}{\Omega}\right)^2 \;=\; \text{larger root of} \quad x^2 \,-\, 3(G+1)x \,+\, G \;=\; 0. \tag{177}$$

For all the TM modes — which all have $G = 0$ — the optimal frequency is $\omega_{\text{opt}} = \sqrt{3} \times \Omega$, while the TE modes have higher $\omega_{\text{opt}}/\Omega$ ratios; for example, the dominant $\text{TE}_{1,0}$ mode of a rectangular waveguide with $a : b = 2 : 1$ has $\omega_{\text{opt}} \approx 2.1 \times \Omega$.

As to the first factor in the equation

$$\alpha(\omega) \;=\; \frac{F(\text{mode})}{aZ_{pw}\sigma\delta(\Omega)} \times \frac{\omega^2 + G(\text{mode})\Omega^2}{\sqrt{\Omega\omega(\omega^2 - \Omega^2)}}\,, \tag{175}$$

it does not depend on the frequency of the actual wave, but it does depend on the frequency band for which the waveguide is designed. If one wants only one mode — such as $\text{TE}_{1,0}$ — to propagate down the waveguide, then one uses a waveguide with

$$1 \;<\; \frac{\omega_{\text{design}}}{\Omega(\text{TE}_{1,0})} \;<\; 2 \tag{178}$$

and hence

$$a \;\sim\; \frac{\pi(c/n)}{\Omega(\text{TE}_{1,0})} \;\sim\; \frac{(1\,\text{to}\,2)\pi(c/n)}{\omega_{\text{design}}}\,. \tag{179}$$

Thus $(1/a) \propto \omega_{\text{design}}$, and also $(1/\delta(\Omega)) \propto \Omega^{1/2} \propto \omega_{\text{design}}^{1/2}$, and consequently

$$\alpha \;\propto\; \left[\omega_{\text{design}}\right]^{3/2}. \tag{180}$$

And that's why the WR-10 waveguide used for the 90 GHz microwaves has 27 times larger attenuation rate than the WR-90 waveguide used for the 10 GHz microwaves, 2.7 dB/m versus 0.1 dB/m.

PS: Eq. (175) gives the attenuation rate due to electric resistivity of the waveguide walls, but there could be additional attenuation due to other causes. For example, for a waveguide filled with a dielectric, the dielectric constant $\epsilon(\omega)$ may develop an imaginary part at high frequencies, which would make the dielectric absorb some of the microwave power and dissipate it as heat. Similarly, a poor dielectric which has a small but non-zero conductivity would absorb some of the microwave power and dissipate it as heat. The air-filled waveguides do not have these kinds of attenuation, but they are vulnerable to corrosion — especially when the air inside them gets dirty and humid — which would drastically decrease the surface resistivity of the metal walls and hence drastically increase the attenuation rate.

Finally, a real-life waveguide may have extra attenuation of the wave power due to scattering by any extraneous objects of size $\gtrsim 0.1\lambda$ inside the waveguide, or any wall deformations of size $\gtrsim 0.1\lambda$. Also, a waveguide which bends too sharply may case wave scattering and hence extra attenuation.

Appendix: F and G Parameters for a Rectangular Waveguide

In this Appendix, I calculate the A, B, C, D, E parameters — and hence F and G — for all the TE and TM modes in a rectangular waveguide. Let's start with a $\mathrm{TM}_{\mathrm{m,n}}$ mode for some integer m and n:

$$\begin{aligned} E_z &= E_0 \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}\,, \\ H_x &= \frac{in\pi\omega\epsilon\epsilon_0}{b\Gamma^2} E_0 \sin\frac{m\pi x}{a} \cos\frac{n\pi y}{b}\,, \\ H_y &= -\frac{im\pi\omega\epsilon\epsilon_0}{a\Gamma^2} E_0 \cos\frac{m\pi x}{a} \sin\frac{n\pi y}{b}\,, \\ \Gamma^2 &= \frac{\pi^2 m^2}{a^2} + \frac{\pi^2 n^2}{b^2}\,. \end{aligned} \tag{181}$$

For this mode

$$B_{m,n} \times a^2 |E_0|^2 \stackrel{\text{def}}{=} \int_0^a dx \int_0^b dy\, |E_z(x,y)|^2 = \frac{ab}{4}\,|E_0|^2, \tag{182}$$

hence

$$B_{m,n} = \frac{ab/4}{a^2} = \frac{b}{4a} \quad \forall m,n. \tag{183}$$

At the same time, along the cross-section's perimeter we have

$$|\mathbf{H}_\parallel|^2 \;=\; \left(\frac{\pi\omega\epsilon\epsilon_0}{\Gamma^2}\right)|E_0|^2 \times \begin{cases} (n/b)^2\sin^2(m\pi x/a) & \text{at the long sides,}\\ (m/a)^2\sin^2(n\pi y/b) & \text{at the short sides,}\end{cases} \tag{184}$$

hence

$$\begin{aligned}\oint\limits_{\text{perimeter}} |\mathbf{H}_\parallel|^2\,d\ell \;&=\; \left(\frac{\pi\omega\epsilon\epsilon_0}{\Gamma^2}\right)|E_0|^2 \times \left(2\times\frac{a}{2}\times(n/b)^2 \;+\; 2\times\frac{b}{2}\times(m/a)^2\right)\\ &=\; \left(\frac{\pi\omega\epsilon\epsilon_0}{\Gamma^2}\right)|E_0|^2 \times \left(\frac{an^2}{b^2}+\frac{bm^2}{a^2}\right).\end{aligned}$$

Interpreting this integral as

$$A_{m,n}\times a\left(\frac{\omega\epsilon\epsilon_0}{\Gamma}\right)^2|E_0|^2, \tag{185}$$

we find

$$A_{m,n} \;=\; \frac{\pi^2}{a\Gamma^2}\left(\frac{an^2}{b^2}+\frac{bm^2}{a^2}\right) \;=\; \frac{(b/a)m^2 \;+\; (a/b)^2n^2}{m^2 \;+\; (a/b)^2n^2}\,. \tag{186}$$

Altogether, for the TM waves we have

$$F(\mathrm{TM}_{m,n}) \;=\; \frac{A_{m,n}}{B_{m,n}} \;=\; 4\,\frac{m^2 \;+\; (a/b)^3n^2}{m^2 \;+\; (a/b)^2n^2}\,, \tag{187}$$

which varies in the range

$$4 \;<\; F(\mathrm{TM}) \;<\; 4(a/b). \tag{188}$$

Now consider a $\mathrm{TE}_{m,n}$ mode

$$\begin{aligned} H_z \;&=\; H_0\,\cos\frac{m\pi x}{a}\,\cos\frac{n\pi y}{b}\,,\\ H_x \;&=\; -\frac{im\pi k}{a\Gamma^2}H_0\,\sin\frac{m\pi x}{a}\,\cos\frac{n\pi y}{b}\,,\\ H_y \;&=\; -\frac{in\pi k}{b\Gamma^2}H_0\,\cos\frac{m\pi x}{a}\,\sin\frac{n\pi y}{b}\,,\\ \Gamma^2 \;&=\; \frac{\pi^2m^2}{a^2} \;+\; \frac{\pi^2n^2}{b^2}\,. \end{aligned} \tag{189}$$

Similarly to what we had earlier for the TM modes, for this TE mode we have

$$E_{m,n} \times a^2|H_0|^2 \stackrel{\text{def}}{=} \int\limits_0^a dx \int\limits_0^b dy\,|H_z(x,y)|^2 \;=\; \frac{ab}{4}\,|H_0|^2, \tag{190}$$

provided both $m > 0$ and n^2 so that both $\cos^2(m\pi x/a)$ and $\cos^2(n\pi y/b)$ average to $\frac{1}{2}$; other wise, for $m = 0$ or $n = 0$ we have one of the $\cos^2$ factors being 1 for all x or all y, hence

$$E_{m,n} \times a^2|H_0|^2 \stackrel{\text{def}}{=} \int\limits_0^a dx \int\limits_0^b dy\,|H_z(x,y)|^2 \;=\; \frac{ab}{2}\,|H_0|^2. \tag{191}$$

Thus,

$$E_{m,n} \;=\; \frac{ab/(4\,\text{or}\,2)}{a^2} \;=\; \frac{b}{a} \times \begin{cases} \frac{1}{4} & \text{for } m > 0 \text{ and } n > 0, \\ \frac{1}{2} & \text{for } m = 0 \text{ or } n = 0. \end{cases} \tag{192}$$

As to the magnetic field at the cross-section's perimeter, we now have

along a long side

$$|\mathbf{H}_\parallel|^2 \;=\; |H_0|^2 \times \cos^2\frac{m\pi x}{a} \;+\; \frac{m^2\pi^2k^2}{a^2\Gamma^4}\,|H_0|^2 \times \sin^2\frac{m\pi x}{a}\,,$$

along a short side

$$|\mathbf{H}_\parallel|^2 \;=\; |H_0|^2 \times \cos^2\frac{n\pi y}{b} \;+\; \frac{n^2\pi^2k^2}{b^2\Gamma^4}\,|H_0|^2 \times \sin^2\frac{n\pi y}{b}\,, \tag{193}$$

hence for $m > 0$ and $n > 0$

$$\begin{aligned}\oint\limits_{\text{perimeter}} |\mathbf{H}_\parallel|^2\,d\eta \;&=\; 2 \times \frac{a}{2}\left(|H_0|^2 + \frac{m^2\pi^2k^2}{a^2\Gamma^4}\,|H_0|^2\right) \;+\; 2 \times \frac{b}{2}\left(|H_0|^2 + \frac{n^2\pi^2k^2}{b^2\Gamma^4}\,|H_0|^2\right) \\ &=\; (a+b) \times |H_0|^2 \;+\; \left(\frac{m^2}{a} + \frac{n^2}{b}\right) \times \frac{\pi^2k^2}{\Gamma^4}\,|H_0|^2,\end{aligned} \tag{194}$$

but for $m = 0$

$$\begin{aligned}\oint\limits_{\text{perimeter}} |\mathbf{H}_\parallel|^2\,d\eta \;&=\; 2 \times a \times |h_0|^2 \;+\; 2 \times \frac{b}{2}\left(|H_0|^2 + \frac{n^2\pi^2k^2}{b^2\Gamma^4}\,|H_0|^2\right) \\ &=\; (2a+b) \times |H_0|^2 \;+\; \frac{n^2}{b} \times \frac{\pi^2k^2}{\Gamma^4}\,|H_0|^2,\end{aligned} \tag{195}$$

and likewise for $n = 0$

$$\oint\limits_{\text{perimeter}} |\mathbf{H}_{\|}|^2 \, d\eta \;=\; (a + 2b) \times |H_0|^2 \;+\; \frac{m^2}{a} \times \frac{\pi^2 k^2}{\Gamma^4} \, |H_0|^2. \tag{196}$$

Interpreting these integrals as

$$C_{m,n} \times a \times |H_0|^2 \;+\; D_{m,n} \times \frac{ak^2}{\Gamma^2} \times |H_0|^2 \,, \tag{197}$$

we find

$$C_{m,n} \;=\; \begin{cases} \frac{a+b}{a} & \text{for } m > 0 \text{ and } n > 0, \\ \frac{2a+b}{a} & \text{for } m = 0 \text{ but } n > 0, \\ \frac{a+2b}{a} & \text{for } n = 0 \text{ but } m > 0, \end{cases} \tag{198}$$

while

$$D_{m,n} \;=\; \frac{\pi^2}{a\Gamma^2} \left(\frac{m^2}{a} + \frac{n^2}{b} \right) \;=\; \frac{m^2 \;+\; (a/b) n^2}{m^2 \;+\; (a/b)^2 n^2} \,. \tag{199}$$

Consequently, the TE waves have

$$\begin{aligned} F(\mathrm{TE}_{m,n}) \;&=\; \frac{D_{m,n}}{E_{m,n}} \;=\; \frac{m^2 \;+\; (a/b) n^2}{m^2 \;+\; (a/b)^2 n^2} \times \frac{a}{b} \times \begin{cases} 4 & \text{for } m > 0 \text{ and } n > 0, \\ 2 & \text{for } m = 0 \text{ or } n = 0, \end{cases} \\ &=\; \begin{cases} 4\, \frac{(a/b)m^2 + (a/b)^2 n^2}{m^2 + (a/b)^2 n^2} & \text{for } m > 0 \text{ and } n > 0, \\ 2 & \text{for } m = 0 \text{ but } n > 0, \\ 2(a/b) & \text{for } n = 0 \text{ but } m > 0, \end{cases} \end{aligned} \tag{200}$$

which varies in the range

$$2 \;\le\; F(\mathrm{TE}) \;< 4(a/b), \tag{201}$$

while

$$G(\mathrm{TE}_{m,n}) \;=\; \frac{C_{m,n}}{D_{m,n}} \;-\; 1 \;=\; \begin{cases} \frac{(b/a)m^2 + (a/b)^2 n^2}{m^2 + (a/b) n^2} & \text{for } m > 0 \text{ and } n > 0, \\ 2(a/b) & \text{for } m = 0 \text{ but } n > 0, \\ 2(b/a) & \text{for } n = 0 \text{ but } m > 0, \end{cases} \tag{202}$$

which varies in the range

$$\frac{b}{a} \;<\; G(\mathrm{TE}) \;\le \frac{2a}{b} \,. \tag{203}$$

In particular, the dominant $\mathrm{TE}_{1,0}$ wave — which is the mode with the lowest cutoff frequency

— has

$$F(\mathrm{TE}_{1,0}) \;=\; \frac{2a}{b}\,, \qquad G(\mathrm{TE}_{1,0}) \;=\; \frac{2b}{a}\,. \tag{204}$$

Resonator Cavities

- Wikipedia article about microwave cavities.

In principle, a microwave cavity can have any kind of geometry. In practice, most cavities are cylinders of radius R and length d of comparable magnitudes; less common are rectangular $a \times b \times d$ cavities. Either way, the cavity can be thought as a finite length d of a waveguide in which some kind of a TM or TE wave travels along the z axis and gets reflected back and forth of the two conducting end-caps at $z = 0$ and $z = d$, thus

$$\mathbf{E}, \mathbf{B} \;\propto\; Ae^{ikz-i\omega t} \;+\; Be^{-ikz-i\omega t}. \tag{205}$$

Approximating the end-caps — as well as the its walls — as perfectly conducting, we also make them perfectly reflecting, thus $|B| = |A|$, while the relative phase of the forward and backward amplitudes A and B is different for the TM and the TE waves.

Indeed, consider a TM wave

$$H_z(x,y,z,t) \;\equiv\; 0, \tag{206}$$

$$E_z(x,y,z,t) \;=\; \psi(x,y)\left(Ae^{+ikz-i\omega t} + Be^{-ikz-i\omega t}\right), \tag{207}$$

$$\mathbf{E}_t(x,y,z,t) \;=\; \frac{ik}{\gamma^2}\,\nabla_t\psi(x,y)\left(Ae^{+ikz-i\omega t} - Be^{-ikz-i\omega t}\right), \tag{208}$$

$$\mathbf{H}_t(x,y,z,t) \;=\; \frac{i\omega\mu\mu_0}{\Gamma^2}\left(\hat{\mathbf{z}} \times \nabla_t\psi(x,y)\right)\left(Ae^{+ikz-i\omega t} + Be^{-ikz-i\omega t}\right). \tag{209}$$

Note the relative signs (marked in red) between the forward and the backward waves here. In particular, note the opposite sign for the $\mathbf{E}_t$ components, which stems from the opposite signs of $\frac{\partial}{\partial z} \to \pm ik$.

At the perfectly conducting endcap at $z = 0$, we have boundary conditions

$$\mathbf{E}_t = 0, \qquad H_z = 0. \tag{210}$$

The $H_z = 0$ condition here is automatic for the TM waves, while the $\mathbf{E}_t = 0$ condition calls for $A - B = 0$. Likewise, similar boundary conditions at the other endcap at $z = d$ call for $Ae^{+ikd} = Be^{-ikd}$. Altogether, this requires

$$k = \frac{p\pi}{d} \quad \text{for a non-negative integer } p = 0, 1, 2, 3, \ldots \tag{211}$$

while

$$E_z = A\cos\frac{p\pi x}{d} \times \psi(x,y)e^{-i\omega t}. \tag{212}$$

On the other hand, a TE wave has

$$E_z(x,y,z,t) \equiv 0, \tag{213}$$

$$H_z(x,y,z,t) = \psi(x,y)\left(Ae^{+ikz-i\omega t} + Be^{-ikz-i\omega t}\right), \tag{214}$$

$$\mathbf{H}_t(x,y,z,t) = \frac{ik}{\gamma^2}\nabla_t\psi(x,y)\left(Ae^{+ikz-i\omega t} - Be^{-ikz-i\omega t}\right), \tag{215}$$

$$\mathbf{E}_t(x,y,z,t) = \frac{-i\omega\epsilon\epsilon_0}{\Gamma^2}\left(\hat{\mathbf{z}}\times\nabla_t\psi(x,y)\right)\left(Ae^{+ikz-i\omega t} + Be^{-ikz-i\omega t}\right). \tag{216}$$

For this wave, both $H_z = 0$ and $\mathbf{E}_t = 0$ boundary conditions at each endcap are non-trivial; however, both conditions lead to the same relations between the forward and backward amplitudes. Specifically

$$\begin{aligned} A + B &= 0 \quad \text{to keep } H_z = 0 \text{ and } \mathbf{E}_t = 0 \text{ at } z = 0, \\ Ae^{+ikd} + Be^{-ikd} &= 0 \quad \text{to keep } H_z = 0 \text{ and } \mathbf{E}_t = 0 \text{ at } z = d, \end{aligned} \tag{217}$$

which together require

$$k = \frac{p\pi}{d} \quad \text{for a positive integer } p = 1, 2, 3, 4, \ldots$$

while

$$H_z = iA\sin\frac{p\pi x}{d} \times \psi(x,y)e^{-i\omega t}. \tag{218}$$

Either way, we have a discrete set of wave numbers k and hence discrete set of resonance frequencies:

$$\frac{\epsilon\mu\omega^2}{c^2} = \Gamma^2 + k^2 = \Gamma^2 + \left(\frac{\pi p}{d}\right)^2 \tag{219}$$

and hence

$$\omega_{m,n,p} = \frac{c}{\sqrt{\epsilon\mu}}\sqrt{\Gamma^2_{m,n} + \left(\frac{\pi p}{d}\right)^2}. \tag{220}$$

For example, a rectangular $a \times b \times d$ cavity resonates at frequencies

$$\omega_{m,n,p} = \frac{c}{\sqrt{\epsilon\mu}}\sqrt{\frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2} + \frac{p^2\pi^2}{d^2}} \tag{221}$$

for integer m, n, p. Moreover, at least two of this integers must be positive while the third may vanish. Specifically, the TM waves must have positive m and n while p may vanish, while the TE waves may have either $m = 0$ or $n = 0$ (but not both) while p must be positive.

The lowest frequency resonance — called the *fundamental mode* — of the rectangular cavity with $d > a > b$ is the $\mathrm{TE}_{1,0,1}$ wave with $m = 1$, $n = 0$, and $p = 1$. For this mode,

$$\omega^2 = \frac{\pi^2 c^2}{\epsilon\mu}\left(\frac{1}{a^2} + \frac{1}{d^2}\right), \tag{222}$$

$$H_z = H_0 \cos\frac{\pi x}{a} \sin\frac{\pi z}{d} e^{-i\omega t}, \tag{223}$$

$$H_x = -\frac{a}{d} H_0 \sin\frac{\pi x}{a} \cos\frac{\pi z}{d} e^{-i\omega t}, \tag{224}$$

$$E_y = i\sqrt{1 + \frac{a^2}{d^2}}\, Z_{pw} H_0 \sin\frac{\pi x}{a} \sin\frac{\pi z}{d} e^{-i\omega t}. \tag{225}$$

$$H_y = E_x = E_z = 0. \tag{226}$$

For a cylindrical cavity, the TM and the TE waves have

$$\Gamma(\mathrm{TM}_{m,n}) = \frac{j_{m,n}}{R}, \qquad \Gamma(\mathrm{TE}_{m,n}) = \frac{j'_{m,n}}{R}, \tag{227}$$

where $j_{m,n}$ is the n^{th} positive root of the m^{th} Bessel function $J_m(x)$ and $j'_{m,n}$ is the n^{th} positive root of its derivative $dJ_m(x)/dx$. Consequently, the resonant frequencies for these waves are

$$\omega^2_{m,n,p}(\mathrm{TM}) = \frac{c^2}{\epsilon\mu}\left(\frac{(j_{m,n})^2}{R^2} + \frac{p^2\pi^2}{d^2}\right)$$
$$\text{for } m = 0,1,2,\dots,\ n = 1,2,3,\dots, p = 0,1,2,\dots, \tag{228}$$

$$\omega^2_{m,n,p}(\mathrm{TE}) = \frac{c^2}{\epsilon\mu}\left(\frac{(j'_{m,n})^2}{R^2} + \frac{p^2\pi^2}{d^2}\right)$$
$$\text{for } m = 0,1,2,\dots,\ n = 1,2,3,\dots, p = 1,2,3\dots. \tag{229}$$

For a stubby cavity with $d < 2.03R$, the fundamental mode is $\mathrm{TM}_{0,1,0}$ with

$$\omega = \frac{c}{\sqrt{\epsilon\mu}} \times \frac{j_{0,1} \approx 2.40}{R},$$
$$E_z(\rho,\phi,z) = E_0\, J_0(j_{0,1}\rho/R), \tag{230}$$
$$H_\phi(\rho,\phi,z) = \frac{iE_0}{Z_{pw}}\, J'_0(j_{0,1}\rho/R), \tag{231}$$
$$E_\rho = E_\phi = 0, \tag{232}$$
$$H_z = H_\rho = 0. \tag{233}$$

But for a longer cavity with $d > 2.03R$, the fundamental mode becomes $\mathrm{TE}_{1,1,1}$ with

$$\omega = \frac{c}{\sqrt{\epsilon\mu}}\sqrt{\frac{(j'_{1,1} \approx 1.84)^2}{R^2} + \frac{\pi^2}{d^2}}, \tag{234}$$

$$H_z = H_0 \times J_1\left(\frac{j'_{1,1}\rho}{R}\right) \times \cos(\phi) \times \sin\frac{\pi z}{d}, \tag{235}$$

$$H_\rho = \frac{\pi R}{j'_{1,1} d}\, H_0 \times J'_1\left(\frac{j'_{1,1}\rho}{R}\right) \times \cos(\phi) \times \cos\frac{\pi z}{d}, \tag{236}$$

$$H_\phi = -\frac{\pi R}{j'_{1,1} d}\, H_0 \times \frac{R}{j'_{1,1}\rho}\, J_1\left(\frac{j'_{1,1}\rho}{R}\right) \times \sin(\phi) \times \cos\frac{\pi z}{d}, \tag{237}$$

$$
\begin{aligned}
E_z &= 0, & (238)\\
E_\rho &= -i\sqrt{1+\left(\frac{\pi R}{j'_{1,1}d}\right)^2}\, Z_{pw}H_0 \times \frac{R}{j'_{1,1}\rho}\, J_1\left(\frac{j'_{1,1}\rho}{R}\right) \times \sin(\phi) \times \sin\frac{\pi z}{d}\,, & (239)\\
E_\phi &= i\sqrt{1+\left(\frac{\pi R}{j'_{1,1}d}\right)^2}\, Z_{pw}H_0 \times J'_1\left(\frac{j'_{1,1}\rho}{R}\right) \times \cos(\phi) \times \sin\frac{\pi z}{d}\,. & (240)
\end{aligned}
$$

Quality of a Resonator Cavity

The quality of any kind of a resonator — be it a mechanical pendulum, an LC circuit, or a microwave cavity, — is defined as

$$
Q \;=\; \omega_0 \times \frac{\text{stored energy}}{\text{power loss}} \tag{241}
$$

where ω_0 is the central frequency of the resonance. In class, I am going to explain this subject off-the-notes.

In these notes, I am going to focus on the microwave cavities and estimate their qualities as resonators. In general, a microwave cavity looses power to 3 mechanisms: (1) Ohmic losses in the cavity walls, (2) losses in the dielectric filling the cavity, and (3) losses through the hole in the wall or the antenna connecting the cavity to the outside world, thus

$$
P^{\text{net}}_{\text{loss}} \;=\; P_1 \;+\; P_2 \;+\; P_3 \tag{242}
$$

and hence

$$
\frac{1}{Q_{\text{net}}} \;=\; \frac{1}{Q_1} \;+\; \frac{1}{Q_2} \;+\; \frac{1}{Q_3}\,. \tag{243}
$$

In these notes, I am going to focus on the first mechanism and estimate Q_1, but in real life the other 2 mechanisms might reduce the net quality of the microwave cavity.

Similarly to the waveguides, the power loss due to Ohmic resistance in the walls is

$$
P \;=\; \frac{R_s}{2}\iint |\mathbf{H}|^2\, d^2\text{area} \tag{244}
$$

where the area integral is over the entire surface of the cavity — including both the side walls and the end caps, — while

$$R_s = \frac{1}{\sigma\delta} = \sqrt{\frac{\omega\mu_{\rm metal}\mu_0}{2\sigma}} \tag{245}$$

is the surface resistivity of the metal; for a copper wall and $\omega = 2\pi \times 10$ GHz, $R_s \approx 26$ mΩ. At the same time, the energy stored in the cavity is

$$U = \iiint\left(\frac{\epsilon\epsilon_0}{4}|\mathbf{E}|^2 + \frac{\mu\mu_0}{4}|\mathbf{H}|^2\right) d^3\mathbf{x} = \frac{\mu\mu_0}{2}\iiint|\mathbf{H}|^2\, d^3\mathbf{x}, \tag{246}$$

hence cavity quality

$$Q = \frac{\omega_0 U}{P} = \frac{\omega\mu\mu_0}{R_s} \times \frac{\iiint|\mathbf{H}|^2\, d^3\text{volume}}{\iint|\mathbf{H}|^2\, d^2\text{area}}\,. \tag{247}$$

Sometimes, this formula is written down as

$$Q = \frac{G}{R_s} \tag{248}$$

where

$$G = \omega_0\mu\mu_0 \times \frac{\iiint|\mathbf{H}|^2\, d^3\text{volume}}{\iint|\mathbf{H}|^2\, d^2\text{area}} \tag{249}$$

is often called the *geometry factor* because it depends on the cavity's geometry but not on the conductivity of it's walls. But just as often, the name *geometry factor* refers to the *dimensionless geometry factor*

$$\hat{G} = \frac{G}{Z_{pw}} = \frac{\omega\sqrt{\epsilon\mu}}{c} \times \frac{\iiint|\mathbf{H}|^2\, d^3\text{volume}}{\iint|\mathbf{H}|^2\, d^2\text{area}}\,; \tag{250}$$

it depends on a particular resonating mode of the cavity as well as on the *ratios* of its dimensions — for example on the d/R ratio of a cylindrical cavity, — but does not depends on its overall size. In terms of this dimensionless geometry factor, the quality factor of the microwave cavity is

$$Q = \frac{Z_{pw}}{R_s} \times \hat{Q}. \tag{251}$$

As an order-of-magnitude estimate,

$$\frac{\iiint |\mathbf{H}|^2 \, d^3\text{volume}}{\iint |\mathbf{H}|^2 \, d^2\text{area}} \sim \frac{\text{cavity's volume}}{\text{cavity's area}} \sim (\text{cavity's size}), \tag{252}$$

while

$$\frac{\omega\sqrt{\epsilon\mu}}{c} \sim \frac{1}{\text{cavity's size}}. \tag{253}$$

hence

$$\hat{G} \sim 1. \tag{254}$$

Thus, a typical microwave cavity has

$$Q \sim \frac{Z_{\text{pw}}}{R_s} \sim 10^4. \tag{255}$$

To conclude this section, let me actually calculate the geometry factor $\hat{G}$ for the fundamental $\text{TM}_{0,1,0}$ mode of a cylindrical cavity with $d < 2.03R$. In this mode, the magnetic field points in the ϕ direction throughout the cavity while its magnitude depends only on the radial coordinate ρ but on on ϕ or z,

$$H_\phi = H_0 \, J_0'(\Gamma\rho) \quad \text{for} \quad \Gamma = \frac{j_{0,1} \approx 2.40}{R}. \tag{256}$$

Consequently,

$$\iint\limits_{\text{sidewall}} |\mathbf{H}|^2 \, d^2\text{area} = 2\pi R d \times |H_0|^2 \times C \tag{257}$$

where

$$C = |J_0'(\Gamma R)|^2 = |J_0'(j_{0,1})|^2 \approx 0.270, \tag{258}$$

while

$$\iint\limits_{\text{endcap}} |\mathbf{H}|^2\, d^2\text{area} \;=\; 2\pi|H_0|^2 \times \int\limits_0^R |J_0'(\Gamma\rho)|^2\,\rho\,d\rho = \; 2\pi R^2 \times |H_0|^2 \times \frac{1}{(j_{0,1})^2} \int\limits_0^{j_{0,1}} (J_0'(x))^2\, x\, dx \tag{259}$$

where

$$\frac{1}{(j_{0,1})^2} \int\limits_0^{j_{0,1}} (J_0'(x))^2\, x\, dx \;=\; \frac{C}{2} \tag{260}$$

for the same constant C as in eq. (258), — which really surprised me when I have calculated this integral using Mathematica, — hence altogether

$$\iint\limits_{\substack{\text{whole}\\ \text{surface}}} |\mathbf{H}|^2\, d^2\text{area} \;=\; 2\pi Rd|H_0|^2 \times C \;+\; 2\times 2\pi R^2|H_0|^2 \times \frac{C}{2} \;=\; 2\pi C\, R(d+R) \times |H_0|^2. \tag{261}$$

At the same time,

$$\iiint |\mathbf{H}|^2\, d^3\text{volume} \;=\; 2\pi d|H_0|^2 \times \int\limits_0^R |J_0'(\Gamma\rho)|^2\,\rho\,d\rho \;=\; 2\pi d|H_0|^2 \times \frac{CR^2}{2}\,, \tag{262}$$

so the ratio of the volume to surface integrals amounts to

$$\iiint |\mathbf{H}|^2\, d^3\text{volume} \Big/ \iint |\mathbf{H}|^2\, d^2\text{area} \;=\; \frac{\pi C d R^2}{2\pi C R(R+d)} \;=\; \frac{Rd}{2(R+d)}\,. \tag{263}$$

Meanwhile the resonating frequency ω_0 of the $\mathrm{TM}_{0,1,0}$ mode is

$$\omega_0 \;=\; \frac{c}{\sqrt{\epsilon\mu}} \times \left(\Gamma = \frac{j_{0,1}}{R}\right), \tag{264}$$

hence

$$\hat{G} \;=\; \frac{j_{0,1}}{R} \times \frac{Rd}{2(R+d)} \;=\; \frac{j_{0,1}}{2} \times \frac{d}{R+d} \;\approx\; \frac{1.20\, d}{R+d}\,. \tag{265}$$

For example, in a cavity with $d = R$ the fundamental mode has $\hat{G} = 0.60$. For a more specific example, consider an air-filled cavity with copper walls and $R = 1.15$ cm which makes for

$\omega_0 = 2\pi \times 10.0$ GHz. At this frequency, copper has $r_s \approx 25.8$ mΩ, hence quality factor

$$Q \;\approx\; \frac{377\ \Omega}{26\ \mathrm{m}\Omega} \times 0.60 \;\approx\; 8800. \tag{266}$$

For a longer cylindrical cavity with $d > 2.03R$, the fundamental mode switches from the $\mathrm{TM}_{0,1,0}$ to the $\mathrm{TE}_{1,1,1}$, and the calculation becomes more complicated. So instead of doing it here, let me put it on your next homework set#10 as problem#4.